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Weber and Beltrami integrals of squared spherical Bessel functions: finite series evaluation and high-index asymptotics

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Weber integrals $\int_0^\infty k^{2+\mu}\mathrm{e}^{-ak^2}j_n^2(pk)\mathrm{d}k$ and Beltrami integrals $\int_0^\infty k^{2+\mu}\mathrm{e}^{-bk}j_n^2(pk)\mathrm{d}k$ are studied, which arise in the multipole expansions of spherical random fields. These integrals define spectral averages of squared spherical Bessel functions j_n^2 with Gaussian or exponentially cut power-law densities. Finite series representations of the integrals are derived for integer power-law index μ, which admit high-precision evaluation at low and moderate Bessel index n. At high n, numerically tractable uniform asymptotic approximations are obtained on the basis of the Debye expansion of modified spherical Bessel functions in the case of Weber integrals. The high-n approximation of Beltrami integrals can be reduced to Legendre asymptotics. The Airy approximation of Weber and Beltrami integrals is derived as well, and numerical tests are performed over a wide range of Bessel indices by comparing the exact finite series expansions of the integrals with their high-index asymptotics. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction

We investigate two classes of Bessel integrals containing squared spherical Bessel functions, which arise in the spectral theory of spherical Gaussian random fields, in multipole expansions of correlation functions of the cosmic microwave background [1, 2]. We study Weber integrals, $\int_0^\infty k^{2+\mu}\mathrm{e}^{-ak^2}j_n^2(pk)\mathrm{d}k$, that is squared spherical Bessel functions averaged with Gaussian power laws, as well as Beltrami integrals, $\int_0^\infty k^{2+\mu}\mathrm{e}^{-bk}j_n^2(pk)\mathrm{d}k$, where the average is performed with exponentially cut power laws at integer power-law exponent μ. We derive finite series expansions of these integrals and also study their high-index asymptotics, obtaining asymptotic approximations suitable for numerical evaluation at high Bessel index n.

In Section 2, we obtain exact finite series representations of Weber integrals, which allow high-precision evaluation at low and moderate Bessel index. In Section 3, we derive the high-index asymptotics of Weber integrals based on the Debye expansion of modified Bessel functions of the first kind.

In Section 4, we study Beltrami integrals with integer power-law exponents $\mu \geq -1$. We derive explicit finite series expansions of these integrals as well as an asymptotic high-n approximation in terms of modified Bessel functions of the second kind. In Section 5, we derive the Airy approximation of Weber and Beltrami integrals valid for real power-law exponents μ.

In Section 6, we investigate Beltrami integrals with negative integer power-law exponent $\mu \leq -2$, which can be reduced to multiple antiderivatives of finite Legendre series. The latter are studied in Section 7 and Appendix A, where finite series representations of the antiderivatives are derived suitable for high-precision evaluation.

In Tables I–VII, we give numerical examples of Weber and Beltrami integrals, testing their asymptotic high-n approximations over a wide range of Bessel indices by comparison with the exact finite series expansions. In Section 8, we present our conclusions.

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2. Weber averages $\int_0^\infty k^{2+\mu} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$

We calculate the integrals

$$E_\mu(n,p;a) = \int_0^\infty k^{2+\mu} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k, \tag{2.1}$$

where the squared spherical Bessel function $j_n(x) = \sqrt{\pi/(2x)}J_{n+1/2}(x)$ [3, 4] is averaged with a Gaussian power law $k^\mu \mathrm{e}^{-ak^2}$, $Re a > 0$, and p is a positive scale parameter. The Bessel index n is a nonnegative integer. For this integral to converge, the condition $\mu + 2 + 2n > -1$ has to be satisfied because $j_n(x) \propto x^n(1 + \mathrm{O}(x^2))$ and $j_n(x) = \mathrm{O}(1/x)$. The factor k^2 in the integrand typically stems from a volume integration [2].

Integral (2.1) is elementary at $\mu = 0$ (Weber's second exponential integral [5,6]),

$$E_0(n,p;a) = \int_0^\infty k^2 \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k = \frac{\sqrt{\pi/2}}{p^3} x^{3/2} \mathrm{e}^{-x} \mathrm{i}^{-n} j_n(\mathrm{i}x), \tag{2.2}$$

where $x = p^2/(2a)$ and $\mathrm{i}^{-n} j_n(\mathrm{i}x) = i_n(x)$ is a modified spherical Bessel function of the first kind, $i_n(x) = \sqrt{\pi/(2x)}I_{n+1/2}(x)$, which admits, for nonnegative integer index n, the finite series expansion [3,5]

$$i_n(x) = \frac{\mathrm{e}^x}{2x} \sum_{k=0}^n (-1)^k \frac{[n,k]}{2^k} \frac{1}{x^k} - (-1)^n \frac{\mathrm{e}^{-x}}{2x} \sum_{k=0}^n \frac{[n,k]}{2^k} \frac{1}{x^k}, \tag{2.3}$$

where we use the Hankel symbol

$$[n,k] = \frac{1}{\Gamma(1+k)} \frac{\Gamma(n+1+k)}{\Gamma(n+1-k)} = \frac{(n+k)!}{k!(n-k)!}. \tag{2.4}$$

Thus, integral (2.2) can be written as finite series,

$$E_0(n,p;a) = \frac{\sqrt{\pi}}{4p^3} \sum_{k=0}^n (-1)^k [n,k] \frac{a^{k-1/2}}{p^{2k-1}} \left(1 - (-1)^{n+k} \mathrm{e}^{-p^2/a}\right), \tag{2.5}$$

which can be used for high-precision evaluation, see Table I.

2.1. Weber integrals with positive power-law exponent

By differentiating the Weber series (2.5) with respect to a, we find

$$E_2(n,p;a) = \int_0^\infty k^4 \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k = -E_0^{(1)}(n,p;a), \tag{2.6}$$

Table I. Weber integral $E_0(n,p;a)$ defined in (2.2), with parameters $p = 1$ and $a = 6.26 \times 10^{-5}$. The integral is calculated at the indicated Bessel index n. In the second column, we use the exact finite series (2.5); the indicated decimal digits are significant, and the numbers are truncated without rounding (in all tables). We also calculated this integral in Debye expansion, see (3.14) and (3.15), with the series coefficients $U_{1,2}$ in (3.19). The second-order Debye expansion coincides with the exact series evaluation in the indicated decimal digits. In the third column, we list the Airy approximation (5.9), whose accuracy is evidently limited. The exponent $a = 6.26 \times 10^{-5}$ is also used in Tables II and III, stemming from a multipole spectral fit of cosmic microwave background temperature fluctuations [2]. $E_\mu(n,p;a) = \int_0^\infty k^{2+\mu} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$.

n	$E_0(n,p;a)$ Weber/Debye	$E_0(n,p;a)$ Airy
0	56.005125970	56.00424
1	55.998114128	55.99723
10	55.620775503	55.61992
100	29.759606324	29.76007
300	$1.9651353936 \times 10^{-1}$	1.964489×10^{-1}
500	$8.7041426233 \times 10^{-6}$	8.668000×10^{-6}
800	$2.1895639884 \times 10^{-16}$	2.122742×10^{-16}
1000	$3.6977148212 \times 10^{-26}$	3.421366×10^{-26}
1500	$5.1457748814 \times 10^{-60}$	3.443998×10^{-60}
2000	$3.1497036162 \times 10^{-107}$	$8.841155 \times 10^{-108}$

with

$$E_0^{(1)}(n,p;a) = \frac{\sqrt{\pi}}{4p^5}\sum_{k=0}^{n}(-1)^k[n,k]\frac{a^{k-3/2}}{p^{2k-3}}\left[\frac{2k-1}{2} - (-1)^{n+k}e^{-p^2/a}\left(\frac{p^2}{a}+\frac{2k-1}{2}\right)\right]. \tag{2.7}$$

The superscript (k) indicates k-fold differentiation with respect to the parameter a, $E_\mu^{(k)} = d^k E_\mu / da^k$, with $E_\mu^{(0)} = E_\mu$. Analogously, we find

$$E_4(n,p;a) = \int_0^\infty k^6 e^{-ak^2} j_n^2(pk)dk = E_0^{(2)}(n,p;a), \tag{2.8}$$

where

$$E_0^{(2)}(n,p;a) = \frac{\sqrt{\pi}}{4p^7}\sum_{k=0}^{n}(-1)^k[n,k]\frac{a^{k-5/2}}{p^{2k-5}}\left\{\frac{(2k-1)(2k-3)}{4} - (-1)^{n+k}e^{-p^2/a}\left[\frac{p^4}{a^2}+(2k-3)\frac{p^2}{a}+\frac{(2k-1)(2k-3)}{4}\right]\right\}. \tag{2.9}$$

More generally, for integer $m \geq 0$,

$$E_{2m}(n,p;a) = \int_0^\infty k^{2m+2} e^{-ak^2} j_n^2(pk)dk = (-1)^m E_0^{(m)}(n,p;a). \tag{2.10}$$

In the limit $a \to 0$, we can neglect exponentially small terms $\propto e^{-p^2/a}$ to find the asymptotic m-fold derivative of series (2.5) as

$$E_0^{(m)}(n,p;a\to 0) \sim \frac{\sqrt{\pi}}{2^{m+2}p^{2m+3}}\sum_{k=0}^{n}(-1)^k[n,k]\frac{a^{k-m-1/2}}{p^{2k-2m-1}} \times (2k-1)(2k-3)\cdots(2k-2m+1). \tag{2.11}$$

The Debye expansion of the integrals (2.10), applicable for large Bessel index n, is derived in Section 3 and the Airy approximation in Section 5. Numerical examples are given in Table II, where the exact series evaluation (2.6)–(2.9) of the integrals $E_{2,4}(n,p;a)$ is compared with their Debye and Airy approximations.

Table II. Weber integrals $E_{2,4}(n,p;a)$ defined in (2.6) and (2.8), with $p=1$, $a=6.26\times10^{-5}$, and Bessel index n. The Weber series evaluation (second and fifth column) is based on the finite series (2.7) and (2.9). The second-order Debye approximations (3.14)–(3.16) (calculated with coefficients $U_{1,2}(q)$ and $V_{1,2}(q)$ in (3.19) and (3.20)) are given in the third and sixth column, and the Airy approximation (5.9) in the fourth and seventh column. At high Bessel index, the Airy approximation fails when the integrals become negligible. The Debye approximation of $E_2(n,p;a)$ is noticeably more accurate than the Airy approximation, see after (5.9). A similar accuracy can be achieved for $E_4(n,p;a)$ by adding a further order ($U_3(q)$ and $V_3(q)$ [3]) to the Debye expansions (3.15) and (3.16).

n	$E_2(n,p;a)$ Weber series	$E_2(n,p;a)$ Debye	$E_2(n,p;a)$ Airy	$E_4(n,p;a)$ Weber series	$E_4(n,p;a)$ Debye	$E_4(n,p;a)$ Airy
0	4.4732528730×10^5	4.4732528976×10^5	4.473322×10^5	$1.0718657044\times10^{10}$	$1.0718028022\times10^{10}$	1.071871×10^{10}
1	4.4738129243×10^5	4.4738129489×10^5	4.473882×10^5	$1.0719104369\times10^{10}$	$1.0718475426\times10^{10}$	1.071916×10^{10}
5	4.4816315730×10^5	4.4816315976×10^5	4.481699×10^5	$1.0725384520\times10^{10}$	$1.0724756683\times10^{10}$	1.072544×10^{10}
10	4.5037444421×10^5	4.5037444666×10^5	4.503807×10^5	$1.0743506740\times10^{10}$	$1.0742882048\times10^{10}$	1.074356×10^{10}
50	5.0307810551×10^5	5.0307810761×10^5	5.030729×10^5	$1.1392408514\times10^{10}$	$1.1391872566\times10^{10}$	1.139249×10^{10}
100	5.3829429720×10^5	5.3829429851×10^5	5.382843×10^5	$1.3533561357\times10^{10}$	$1.3533227761\times10^{10}$	1.353333×10^{10}
500	2.2480454163	2.2480454167	2.240569	5.8178391152×10^5	5.8178381840×10^5	5.80265×10^5
1000	$3.7169966942\times10^{-20}$	$3.7169966944\times10^{-20}$	3.45211×10^{-20}	$3.7373005512\times10^{-14}$	$3.7373005172\times10^{-14}$	3.48357×10^{-14}
2000	$1.2440039667\times10^{-100}$	$1.2440039667\times10^{-100}$	3.54529×10^{-101}	$4.9139385782\times10^{-94}$	$4.9139385768\times10^{-94}$	1.42166×10^{-94}

2.2. Integral $\int_0^\infty \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$

We start with the finite Weber series $E_0(n,p;a)$ in (2.5), and perform term-by-term integration with respect to a,

$$E_0^{(-1)}(n,p;a) = \int_0^a E_0(n,p;a)\mathrm{d}a, \tag{2.12}$$

to obtain

$$E_0^{(-1)}(n,p;a) = \frac{\sqrt{\pi}}{2p}\sum_{k=0}^{n}(-1)^k[n,k]\left(\frac{1}{2k+1}\frac{a^{k+1/2}}{p^{2k+1}} - (-1)^{n+k}\frac{1}{2}\Gamma(-k-1/2,p^2/a)\right), \tag{2.13}$$

where $[n,k]$ is the Hankel symbol (2.4), and Γ denotes the incomplete gamma function [3]

$$p^{2\beta+2}\Gamma(-\beta-1,p^2/a) = \int_0^a x^\beta \mathrm{e}^{-p^2/x}\mathrm{d}x. \tag{2.14}$$

Apparently, $\mathrm{d}E_0^{(-1)}/\mathrm{d}a = E_0$, which can readily be checked by way of the identity $\mathrm{d}\Gamma(\alpha,y)/\mathrm{d}y = -y^{\alpha-1}\mathrm{e}^{-y}$.

On the other hand, see (2.2) and (2.12),

$$E_0^{(-1)}(n,p;a) = \int_0^a\int_0^\infty k^2\mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k\mathrm{d}a, \tag{2.15}$$

where we substitute

$$\int_0^a \mathrm{e}^{-ak^2}\mathrm{d}a = -\frac{1}{k^2}\mathrm{e}^{-ak^2} + \frac{1}{k^2}, \tag{2.16}$$

to find

$$E_0^{(-1)}(n,p;a) = -\int_0^\infty \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k + \int_0^\infty j_n^2(pk)\mathrm{d}k. \tag{2.17}$$

By making use of the Schafheitlin integral [6]

$$\int_0^\infty j_n^2(pk)\mathrm{d}k = \frac{\pi}{2p}\frac{1}{2n+1}, \tag{2.18}$$

we obtain

$$E_{-2}(n,p;a) = \int_0^\infty \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k = \frac{\pi}{2p}\frac{1}{2n+1} - E_0^{(-1)}(n,p;a), \tag{2.19}$$

where $E_0^{(-1)}$ is the finite series (2.13). This series representation of integral $E_{-2}(n,p;a)$ is compared with its Airy approximation (5.9) in Table III.

2.3. Integral $\int_0^\infty \frac{1}{k^2}\mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$

This integral is calculated by iterating the integration in Section 2.2, starting with

$$E_0^{(-2)}(n,p;a) = \int_0^a E_0^{(-1)}(n,p;a)\mathrm{d}a, \tag{2.20}$$

where we substitute series $E_0^{(-1)}$ in (2.13). In the term-by-term integration of (2.20), we need some properties of incomplete gamma functions, namely the derivative stated after (2.14), the limit $\Gamma(\alpha,y\to\infty) \sim y^{\alpha-1}\mathrm{e}^{-y}$, the integral representation $b^{-\alpha}\Gamma(\alpha,by) = \int_y^\infty x^{\alpha-1}\mathrm{e}^{-bx}\mathrm{d}x$, as well as identity (2.22). To derive the latter, we apply partial integration to $\mathrm{d}(\Gamma(\alpha,x)x^\beta)/\mathrm{d}x$, to obtain

$$\int_y^\infty \Gamma(\alpha,x)x^{\beta-1}\mathrm{d}x = \frac{1}{\beta}(\Gamma(\alpha+\beta,y) - \Gamma(\alpha,y)y^\beta). \tag{2.21}$$

Thus,

$$\begin{aligned}\int_0^a a^\lambda\Gamma(\alpha,p^2/a)\mathrm{d}a &= p^{2\lambda+2}\int_{p^2/a}^\infty \Gamma(\alpha,y)y^{-\lambda-2}\mathrm{d}y \\ &= \frac{p^{2\lambda+2}}{\lambda+1}\left[\Gamma(\alpha,p^2/a)\frac{a^{\lambda+1}}{p^{2\lambda+2}} - \Gamma(\alpha-\lambda-1,p^2/a)\right].\end{aligned} \tag{2.22}$$

Table III. Weber integrals $E_{-2,-4,-6}(n,p;a)$ defined in (2.19), (2.30), and (2.39), with $p=1$ and $a=6.26\times10^{-5}$. The series evaluation of $E_{-2,-4,-6}$ in columns 2, 4, and 6 is based on the finite Weber series (2.13), (2.25), and (2.34). The integrals E_{-4} and E_{-6} converge for Bessel index $n\geq1$ and $n\geq2$, respectively. The corresponding Airy approximation (5.9) is recorded in columns 3, 5, and 7.

n	$E_{-2}(n,p;a)$ Weber series	$E_{-2}(n,p;a)$ Airy	$E_{-4}(n,p;a)$ Weber series	$E_{-4}(n,p;a)$ Airy	$E_{-6}(n,p;a)$ Weber series	$E_{-6}(n,p;a)$ Airy
0	1.5637844850	1.5637845	–	–	–	–
1	5.1658722645 $\times10^{-1}$	5.1658726 $\times10^{-1}$	2.0940702557 $\times10^{-1}$	1.1632279 $\times10^{-1}$	–	–
5	1.3579221140 $\times10^{-1}$	1.3579224 $\times10^{-1}$	2.4323731914 $\times10^{-3}$	2.3516783 $\times10^{-3}$	6.9590891278 $\times10^{-5}$	5.8372972 $\times10^{-5}$
10	6.7804045235 $\times10^{-2}$	6.7804081 $\times10^{-2}$	3.3794305164 $\times10^{-4}$	3.3483798 $\times10^{-4}$	2.3951794985 $\times10^{-6}$	2.2865785 $\times10^{-6}$
50	8.8965004072 $\times10^{-3}$	8.8965216 $\times10^{-3}$	2.3604075600 $\times10^{-6}$	2.3592108 $\times10^{-6}$	7.3085009297 $\times10^{-10}$	7.2916395 $\times10^{-10}$
100	2.0380652885 $\times10^{-3}$	2.0380601 $\times10^{-3}$	1.5779632936 $\times10^{-7}$	1.5775794 $\times10^{-7}$	1.3104222422 $\times10^{-11}$	1.3092397 $\times10^{-11}$
500	3.3763255799 $\times10^{-11}$	3.3591110 $\times10^{-11}$	1.3118712326 $\times10^{-16}$	1.3037743 $\times10^{-16}$	5.1051358369 $\times10^{-22}$	5.0674595 $\times10^{-22}$
1000	3.6794336235 $\times10^{-32}$	3.3913030 $\times10^{-32}$	3.6621333274 $\times10^{-38}$	3.3619006 $\times10^{-38}$	3.6457956125 $\times10^{-44}$	3.3331348 $\times10^{-44}$
2000	7.9757893623 $\times10^{-114}$	2.2048012 $\times10^{-114}$	2.0199187776 $\times10^{-120}$	5.4983609 $\times10^{-121}$	5.1162359154 $\times10^{-127}$	1.3711987 $\times10^{-127}$

We employ this integral in term-by-term integrations with integer $\lambda\geq0$ and half-integer α. With these prerequisites, integral (2.20) can readily be calculated as

$$E_0^{(-2)}(n,p;a)=\frac{\sqrt{\pi}}{2}p\sum_{k=0}^{n}(-1)^k[n,k]\left\{\frac{2}{(2k+1)(2k+3)}\frac{a^{k+3/2}}{p^{2k+3}}-(-1)^{n+k}\frac{1}{2}\Gamma(-k-1/2,p^2/a)\frac{a}{p^2}\right.$$
$$\left.+(-1)^{n+k}\frac{1}{2}\Gamma(-k-3/2,p^2/a)\right\}. \tag{2.23}$$

The recursive relation

$$\Gamma(\alpha-1,x)=\frac{\Gamma(\alpha,x)}{\alpha-1}-\frac{x^{\alpha-1}e^{-x}}{\alpha-1} \tag{2.24}$$

can be used to eliminate one of the gamma functions in (2.23),

$$E_0^{(-2)}(n,p;a)=\frac{\sqrt{\pi}}{2}p\sum_{k=0}^{n}(-1)^k[n,k]$$
$$\times\left\{\frac{1}{2k+3}\frac{a^{k+3/2}}{p^{2k+3}}\left(\frac{2}{2k+1}+(-1)^{k+n}e^{-p^2/a}\right)\right.$$
$$\left.-(-1)^{n+k}\Gamma(-k-1/2,p^2/a)\left(\frac{1}{2k+3}+\frac{1}{2}\frac{a}{p^2}\right)\right\}. \tag{2.25}$$

On the other hand, we may write integral (2.20) as, see (2.15),

$$E_0^{(-2)}(n,p;a)=\int_0^a\int_0^a\int_0^\infty k^2e^{-ak^2}j_n^2(pk)dk(da)^2. \tag{2.26}$$

Here, we use

$$\int_0^a\int_0^a e^{-ak^2}(da)^2=\frac{1}{k^4}e^{-ak^2}-\frac{1}{k^4}+\frac{a}{k^2}, \tag{2.27}$$

to find

$$E_0^{(-2)}(n,p;a)=\int_0^\infty\frac{1}{k^2}e^{-ak^2}j_n^2(pk)dk-\int_0^\infty\frac{1}{k^2}j_n^2(pk)dk+a\int_0^\infty j_n^2(pk)dk. \tag{2.28}$$

Substituting the Schafheitlin integral (2.18) and

$$\int_0^\infty \frac{1}{k^2} j_n^2(pk)\mathrm{d}k = \frac{p\pi}{(2n+3)(2n+1)(2n-1)}, \tag{2.29}$$

we arrive at

$$\begin{aligned} E_{-4}(n,p;a) &= \int_0^\infty \frac{1}{k^2} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k \\ &= \frac{p\pi}{(2n+3)(2n+1)(2n-1)} - \frac{1}{2}\frac{p\pi}{2n+1}\frac{a}{p^2} + E_0^{(-2)}(n,p;a). \end{aligned} \tag{2.30}$$

The finite series $E_0^{(-2)}$ is stated in (2.25) and tested in Table III by comparison with the Airy approximation of integral $E_{-4}(n,p;a)$.

2.4. Integral $\int_0^\infty \frac{1}{k^4} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$

The leading order of the ascending series expansion of $j_n(x)$ is x^n [3], so that integral $E_{-6}(n,p;a)$ in (2.1) is convergent for Bessel indices $n \geq 2$. We iterate the integration in Sections 2.2 and 2.3, starting with

$$E_0^{(-3)}(n,p;a) = \int_0^a E_0^{(-2)}(n,p;a)\mathrm{d}a, \tag{2.31}$$

where we substitute series (2.23) for $E_0^{(-2)}$. The term-by-term integration is carried out by means of identity (2.22),

$$\begin{aligned} E_0^{(-3)}(n,p;a) = &\frac{\sqrt{\pi}}{2} p^3 \sum_{k=0}^{n} (-1)^k [n,k] \\ &\times \left\{ \frac{4}{(2k+1)(2k+3)(2k+5)} \frac{a^{k+5/2}}{p^{2k+5}} - (-1)^{n+k}\frac{1}{4}\Gamma(-k-1/2,p^2/a)\frac{a^2}{p^4} \right. \\ &\left. +(-1)^{n+k}\frac{1}{2}\Gamma(-k-3/2,p^2/a)\frac{a}{p^2} - (-1)^{n+k}\frac{1}{4}\Gamma(-k-5/2,p^2/a) \right\}. \end{aligned} \tag{2.32}$$

As in (2.25), this can be reduced to one incomplete gamma function in each term via the recursive relations (2.24) and

$$\Gamma(\alpha-2,x) = \frac{\Gamma(\alpha,x)}{(\alpha-1)(\alpha-2)} - \frac{x^{\alpha-1}\mathrm{e}^{-x}}{(\alpha-1)(\alpha-2)} - \frac{x^{\alpha-2}\mathrm{e}^{-x}}{\alpha-2}. \tag{2.33}$$

We thus find

$$\begin{aligned} E_0^{(-3)}(n,p;a) = &\frac{\sqrt{\pi}}{2} p^3 \sum_{k=0}^{n} (-1)^k [n,k] \\ &\times \left\{ \frac{1}{(2k+3)(2k+5)} \frac{a^{k+3/2}}{p^{2k+3}} \left[\frac{4}{2k+1}\frac{a}{p^2} + (-1)^{n+k}\mathrm{e}^{-p^2/a}\left(1+\frac{2k+7}{2}\frac{a}{p^2}\right)\right] \right. \\ &\left. -(-1)^{n+k}\Gamma(-k-1/2,p^2/a)\left(\frac{1}{(2k+3)(2k+5)} + \frac{1}{2k+3}\frac{a}{p^2} + \frac{1}{4}\frac{a^2}{p^4}\right)\right\}. \end{aligned} \tag{2.34}$$

On the other hand, we may write integral (2.31) as, see (2.26),

$$E_0^{(-3)}(n,p;a) = \int_0^a\int_0^a\int_0^a\int_0^\infty k^2\mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k(\mathrm{d}a)^3, \tag{2.35}$$

where we substitute

$$\int_0^a\int_0^a\int_0^a \mathrm{e}^{-ak^2}(\mathrm{d}a)^3 = -\frac{1}{k^6}\mathrm{e}^{-ak^2} + \frac{1}{k^6} - \frac{a}{k^4} + \frac{a^2}{2k^2}, \tag{2.36}$$

to obtain

$$E_0^{(-3)}(n,p;a) = -\int_0^\infty \frac{1}{k^4}\mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k + \int_0^\infty \frac{1}{k^4} j_n^2(pk)\mathrm{d}k - a\int_0^\infty \frac{1}{k^2} j_n^2(pk)\mathrm{d}k + \frac{a^2}{2}\int_0^\infty j_n^2(pk)\mathrm{d}k. \tag{2.37}$$

Here, we substitute the Schafheitlin integrals (2.18), (2.29), and

$$\int_0^\infty \frac{1}{k^4} j_n^2(pk)\mathrm{d}k = \frac{3\pi p^3}{(2n+5)(2n+3)(2n+1)(2n-1)(2n-3)}, \tag{2.38}$$

to arrive at

$$E_{-6}(n,p;a)=\int_0^\infty \frac{1}{t^4}\mathrm{e}^{-at^2}j_n^2(pt)\mathrm{d}t$$
$$=\frac{3\pi p^3}{(2n+5)(2n+3)(2n+1)(2n-1)(2n-3)} \tag{2.39}$$
$$-\frac{\pi p^3}{(2n+3)(2n+1)(2n-1)}\frac{a}{p^2}+\frac{\pi}{4}\frac{p^3}{2n+1}\frac{a^2}{p^4}-E_0^{(-3)}(n,p;a),$$

where $E_0^{(-3)}$ is the finite series calculated in (2.34). As mentioned, this is valid for integer index $n \geq 2$. Numerical examples of the integrals $E_{-2,-4,-6}$ studied in Sections 2.2–2.4 are given in Table III.

2.5. Consistency checks for Weber series

The Weber integrals (2.1) satisfy

$$\frac{\mathrm{d}^k}{\mathrm{d}a^k}E_\mu(n,p;a)=(-1)^k E_{\mu+2k}(n,p;a). \tag{2.40}$$

We have calculated these integrals for $\mu = 4$, see (2.8), $\mu = 2$, see (2.6), $\mu = 0$, see (2.5), $\mu = -2$, see (2.19), $\mu = -4$, see (2.30), and $\mu = -6$, see (2.39). The first consistency check is thus obtained by substituting the respective series into (2.40).

Analogously, we may consider multiple antiderivatives of the Weber series $E_0(n,p;a)$ in (2.5),

$$E_0^{(-k)}(n,p;a)=\int_0^a\cdots\int_0^a E_0(n,p;a)(\mathrm{d}a)^k, \tag{2.41}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}a}E_0^{(-k)}(n,p;a)=E_0^{(1-k)}(n,p;a), \tag{2.42}$$

valid for integer k. Here, we substitute the series $E_0^{(j)}(n,p;a)$ calculated for $j = 2$ in (2.9), $j = 1$ in (2.7), $j = 0$ in (2.5), $j = -1$ in (2.13), $j = -2$ in (2.25), and $j = -3$ in (2.34). The incomplete gamma functions occurring in these series can be reduced to the complementary error function, by way of the recursive relation

$$\Gamma(-k-1/2,x)=\Gamma(-k-1/2)\left[\operatorname{erfc}\left(\sqrt{x}\right)-\frac{\mathrm{e}^{-x}}{x^{k+1/2}}\sum_{j=0}^{k}\frac{x^j}{\Gamma(j-k+1/2)}\right], \tag{2.43}$$

valid for integer $k \geq -1$, where $\operatorname{erfc}\left(\sqrt{x}\right)=\pi^{-1/2}\Gamma(1/2,x)$.

3. High-index asymptotics of Weber integrals

3.1. Weber's integral representation of modified spherical Bessel functions of the first kind

To derive the high-n Debye expansion of the Weber integrals in (2.10), we start with integral (2.2),

$$E_0(n,p;a)=\int_0^\infty k^2\mathrm{e}^{-ak^2}j_n^2(pk)\mathrm{d}k=\frac{\sqrt{\pi/2}}{p^3}e_n(x), \tag{3.1}$$

where we use the shortcuts

$$e_n(x)=x^{3/2}\mathrm{e}^{-x}i_n(x),\quad x=p^2/(2a). \tag{3.2}$$

The modified spherical Bessel function $i_n(x)$ is defined as

$$i_n(x)=\mathrm{i}^{-n}j_n(\mathrm{i}x)=\sqrt{\frac{\pi}{2x}}I_{n+1/2}(x), \tag{3.3}$$

where $I_{n+1/2}(x)$ is a modified Bessel function of half-integer index [3, 6]. The derivative of the rescaled Bessel function $e_n(x)$ in (3.2) reads

$$e_n'(x)=-x^{3/2}\mathrm{e}^{-x}\left[\left(1-\frac{3}{2}\frac{1}{x}\right)i_n(x)-i_n'(x)\right]. \tag{3.4}$$

By making use of the differential equation for modified spherical Bessel functions,

$$i_n''(x) = \left(\frac{n(n+1)}{x^2} + 1\right) i_n(x) - \frac{2}{x} i_n'(x), \tag{3.5}$$

we find

$$e_n''(x) = 2x^{3/2} \mathrm{e}^{-x} \left[\left(1 - \frac{3}{2}\frac{1}{x} + \frac{4n(n+1)+3}{8}\frac{1}{x^2}\right) i_n(x) - \left(1 - \frac{1}{2}\frac{1}{x}\right) i_n'(x) \right]. \tag{3.6}$$

We also note

$$\frac{\mathrm{d}}{\mathrm{d}a} e_n(x) = -\frac{2}{p^2} x^2 e_n'(x), \quad \frac{\mathrm{d}^2}{\mathrm{d}a^2} e_n(x) = \frac{4}{p^4} x^4 \left(e_n''(x) + \frac{2}{x} e_n'(x)\right), \tag{3.7}$$

where $x = p^2/(2a)$, see (3.2).

With these prerequisites, we obtain the first two derivatives $E_0^{(k)} = \mathrm{d}^k E_0/\mathrm{d}a^k$ of Weber's integral (3.1) as

$$E_0^{(1)}(n, p; a) = -\frac{\sqrt{2\pi}}{p^5} x^2 e_n'(x), \tag{3.8}$$

$$E_0^{(2)}(n, p; a) = \frac{\sqrt{8\pi}}{p^7} x^4 \left(e_n''(x) + \frac{2}{x} e_n'(x)\right), \tag{3.9}$$

where we substitute e_n' in (3.4) and e_n'' in (3.6). This is used to set up the Debye expansion of the integrals in (2.10),

$$E_2(n, p; a) = \int_0^\infty k^4 \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k = -E_0^{(1)}(n, p; a), \tag{3.10}$$

$$E_4(n, p; a) = \int_0^\infty k^6 \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k = E_0^{(2)}(n, p; a). \tag{3.11}$$

When applying Debye asymptotics, it is convenient to express the spherical functions occurring in (3.4) and (3.6) by ordinary modified Bessel functions as in (3.3) for $i_n(x)$; the derivative of $i_n(x)$ reads

$$i_n'(x) = \sqrt{\frac{\pi}{2x}} \left(I_{n+1/2}'(x) - \frac{1}{2x} I_{n+1/2}(x)\right). \tag{3.12}$$

We thus find, see (3.2), (3.4), and (3.6),

$$\begin{aligned} e_n(x) &= \sqrt{\frac{\pi}{2}} x \mathrm{e}^{-x} I_{n+1/2}(x), \\ e_n'(x) &= -\sqrt{\frac{\pi}{2}} x \mathrm{e}^{-x} \left[\left(1 - \frac{1}{x}\right) I_{n+1/2}(x) - I_{n+1/2}'(x) \right], \\ e_n''(x) &= \sqrt{2\pi} x \mathrm{e}^{-x} \left[\left(1 - \frac{1}{x} + \frac{(n+1/2)^2}{2}\frac{1}{x^2}\right) I_{n+1/2}(x) - \left(1 - \frac{1}{2}\frac{1}{x}\right) I_{n+1/2}'(x) \right]. \end{aligned} \tag{3.13}$$

3.2. Uniform Debye approximation of Weber integrals

We write the Weber integrals in (3.1), (3.10), and (3.11) as modified Bessel functions,

$$\begin{aligned} E_0(n, p; a) &= \frac{\pi}{2p^3} x \mathrm{e}^{-x} I_{n+1/2}(x), \\ E_2(n, p; a) &= -\frac{\pi}{p^5} x^3 \mathrm{e}^{-x} \left[\left(1 - \frac{1}{x}\right) I_{n+1/2}(x) - I_{n+1/2}'(x) \right], \\ E_4(n, p; a) &= \frac{4\pi}{p^7} x^5 \mathrm{e}^{-x} \\ &\quad \times \left[\left(1 - \frac{2}{x} + \frac{(n+1/2)^2+2}{2}\frac{1}{x^2}\right) I_{n+1/2}(x) - \left(1 - \frac{3}{2}\frac{1}{x}\right) I_{n+1/2}'(x) \right], \end{aligned} \tag{3.14}$$

where $x = p^2/(2a)$, and we used (3.8), (3.9), and (3.13). We introduce a new variable, writing $x = (n+1/2)y$. The high-n Debye expansion of $I_{n+1/2}((n+1/2)y)$ and its derivative $I_{n+1/2}'((n+1/2)y)$ reads [3]

$$I_{n+1/2}(x) \sim \frac{\mathrm{e}^{(n+1/2)\eta}}{\sqrt{2\pi}} \frac{q^{1/2}}{\sqrt{n+1/2}} \left[1 + \frac{U_1(q)}{n+1/2} + \frac{U_2(q)}{(n+1/2)^2} + \cdots\right], \tag{3.15}$$

$$I'_{n+1/2}(x) \sim \frac{e^{(n+1/2)\eta}}{\sqrt{2\pi y}} \frac{1}{q^{1/2}\sqrt{n+1/2}} \left[1 + \frac{V_1(q)}{n+1/2} + \frac{V_2(q)}{(n+1/2)^2} + \cdots \right], \tag{3.16}$$

where

$$q = \frac{1}{\sqrt{1+y^2}}, \quad y = \frac{x}{(n+1/2)} = \frac{p^2}{2a(n+1/2)}, \tag{3.17}$$

with $Rea > 0$, see (3.1), and

$$\eta = \frac{1}{q} + \log \frac{y}{1+1/q} = \frac{1}{q} - \operatorname{arcsinh}\frac{1}{y}. \tag{3.18}$$

The first two expansion coefficients in the asymptotic series of $I_{n+1/2}((n+1/2)y)$ in (3.15) are

$$U_1(q) = \frac{1}{24}\left(3q - 5q^3\right), \quad U_2(q) = \frac{1}{1152}\left(81q^2 - 462q^4 + 385q^6\right), \tag{3.19}$$

and the coefficients of the derivative $I'_{n+1/2}(x)$ (at $x = (n+1/2)y$) indicated in (3.16) read

$$V_1(q) = \frac{1}{24}\left(-9q + 7q^3\right), \quad V_2(q) = \frac{1}{1152}\left(-135q^2 + 594q^4 - 455q^6\right). \tag{3.20}$$

The Debye expansion of the Weber integrals E_0, E_2, and E_4 in (3.1), (3.10), and (3.11) is assembled by replacing the modified Bessel functions in (3.14) by the asymptotic series (3.15) and (3.16). Numerical tests of the Debye approximation of these integrals are performed in Tables I and II.

4. Squared spherical Bessel functions averaged with exponentially cut power laws

4.1. Beltrami integrals and finite Legendre series

We replace the Gaussian factor in the integrand (2.1) by an exponential cutoff and consider the integrals

$$H_\mu(n,p;b) = \int_0^\infty k^{2+\mu} e^{-bk} j_n^2(pk)\,dk, \tag{4.1}$$

where $j_n(x)$ is a spherical Bessel function, p a positive scale parameter, and $k^\mu e^{-bk}$ an exponentially cut and possibly modulated power-law distribution. The exponent μ is real, and b is complex with $Reb > 0$. The Bessel index n is a nonnegative integer. $H_\mu(n,p;b)$ is convergent for $\mu + 2n > -3$ because the ascending series of $j_n(x)$ starts with x^n.

For $\mu = -1$, the Beltrami integral (4.1) is elementary [5],

$$H_{-1}(n,p;b) = \int_0^\infty k e^{-bk} j_n^2(pk)\,dk = \frac{1}{2p^2} Q_n\left(1 + \frac{b^2}{2p^2}\right). \tag{4.2}$$

$Q_n(x)$ is a Legendre function of the second kind with branch cut $[-1,1]$, see [3], which is elementary because of the integer degree $n \geq 0$,

$$\begin{aligned} Q_n(x) =& P_n(x)\left(\frac{1}{2}\log\frac{x+1}{x-1} - \gamma_E - \psi(n+1)\right) \\ &+ \sum_{k=0}^{n} \frac{\Gamma(1+n+k)(\gamma_E + \psi(k+1))}{\Gamma(1+n-k)\Gamma^2(1+k)} \frac{(x-1)^k}{2^k}, \end{aligned} \tag{4.3}$$

where P_n is a Legendre polynomial,

$$P_n(x) = \sum_{k=0}^{n} \frac{\Gamma(1+n+k)}{\Gamma(1+n-k)\Gamma^2(1+k)} \frac{(x-1)^k}{2^k} = \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} (-1)^k \frac{(1-x)^k}{2^k}, \tag{4.4}$$

and $[n,k]$ denotes the Hankel symbol (2.4). In representation (4.4), P_n is a hypergeometric polynomial [6],

$$P_n(x) = {}_2F_1(-n, n+1; 1; (1-x)/2) = \sum_{k=0}^{n} \frac{(-n)_k (1+n)_k}{(1)_k k!} \frac{(1-x)^k}{2^k}, \tag{4.5}$$

where we use the Pochhammer symbol $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, $(\alpha)_0 = 1$, or

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad (-n)_k = \frac{(-1)^k \Gamma(1+n)}{\Gamma(1+n-k)}. \tag{4.6}$$

Table IV. Beltrami integrals $H_{-1}(n,p;b)$ and $H_0(n,p;b)$ as defined in (4.1), with parameters $p=1$ and $b=2.1\times10^{-4}$, used in the spectral fit of the cosmic microwave background in [2]. n denotes the Bessel index. The Legendre series evaluation is compared with the asymptotic modified-Bessel and Airy approximations in Sections 4.3 and 5. In columns 2 and 5, we list the exact series evaluation, see (4.8) and (4.12), based on the finite Legendre series (4.9) and (4.13). All decimal digits are significant, and the numbers are truncated. In columns 3 and 6, we record the modified Bessel approximation, see (4.21) and (4.22), and in columns 4 and 7, the Airy approximation (5.10). The Airy and modified Bessel approximations are high-index approximations but quite efficient at low n as well. $H_\mu(n,p;b)=\int_0^\infty k^{2+\mu}\mathrm{e}^{-bk}j_n^2(pk)\mathrm{d}k$.

n	$H_{-1}(n,p;b)$ Legendre	$H_{-1}(n,p;b)$ modified Bessel	$H_{-1}(n,p;b)$ Airy	$H_0(n,p;b)$ Legendre	$H_0(n,p;b)$ modified Bessel	$H_0(n,p;b)$ Airy
0	4.5807751066	4.6387408597	4.63874087	2.3809523547 $\times10^3$	2.3809522351 $\times10^3$	2.38095225 $\times10^3$
1	4.0807752076	4.0894348171	4.08943483	2.3809514452 $\times10^3$	2.3809513382 $\times10^3$	2.38095135 $\times10^3$
5	3.4391097428	3.4397945275	3.43979453	2.3809407340 $\times10^3$	2.3809406437 $\times10^3$	2.38094066 $\times10^3$
10	3.1162953653	3.1164840281	3.11648403	2.3809134740 $\times10^3$	2.3809133922 $\times10^3$	2.38091340 $\times10^3$
50	2.3312520329	2.3312601991	2.33126020	2.3802612346 $\times10^3$	2.3802611734 $\times10^3$	2.38026119 $\times10^3$
100	1.9873632996	1.9873653601	1.98736536	2.3785797704 $\times10^3$	2.3785797182 $\times10^3$	2.37857973 $\times10^3$
500	1.1890188930	1.1890189748	1.18901897	2.3431592610 $\times10^3$	2.3431592298 $\times10^3$	2.34315924 $\times10^3$
1000	8.5285473453 $\times10^{-1}$	8.5285475434 $\times10^{-1}$	8.52854756 $\times10^{-1}$	2.2657436410 $\times10^3$	2.2657436186 $\times10^3$	2.26574363 $\times10^3$
2000	5.3594972330 $\times10^{-1}$	5.3594972780 $\times10^{-1}$	5.35949728 $\times10^{-1}$	2.0588865983 $\times10^3$	2.0588865842 $\times10^3$	2.05888659 $\times10^3$
10^4	5.0385426695 $\times10^{-2}$	5.0385426748 $\times10^{-2}$	5.03854266 $\times10^{-2}$	6.1367914702 $\times10^2$	6.1367914569 $\times10^2$	6.13679148 $\times10^2$
10^5	–	1.0307732371 $\times10^{-10}$	1.03077320 $\times10^{-10}$	–	1.0550414064 $\times10^{-5}$	1.05504137 $\times10^{-5}$

The constant γ_E in the Legendre function (4.3) is arbitrary because it apparently drops out, and we choose it as Euler's constant, to chancel the Euler constant in the psi function, which reads, at positive integers,

$$\psi(k+1)=-\gamma_E+1+\frac{1}{2}+\ldots+\frac{1}{k},\quad \psi(1)=-\gamma_E. \tag{4.7}$$

We perform a variable change in the Legendre function (4.3), defining $q_n(x)=Q_n(1+2x^2)$, so that integral (4.2) reads

$$H_{-1}(n,p;b)=\frac{1}{2p^2}q_n\left(\frac{b}{2p}\right), \tag{4.8}$$

where

$$q_n(x)=\left(\frac{1}{2}\log(1+x^2)-\log x\right)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}x^{2k}$$

$$+\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}(\psi(k+1)-\psi(n+1))x^{2k}. \tag{4.9}$$

We also define, for nonnegative integers n and k, the shortcut

$$\sigma(n,k)=\psi(n+1)-\psi(k+1)=\sum_{j=k+1}^{n}\frac{1}{j}, \tag{4.10}$$

so that $\sigma(n,k)=0$ for $n\le k$. We use the customary convention that a sum is void if the lower summation boundary exceeds the upper one. In (4.9), we can replace the difference $\psi(k+1)-\psi(n+1)$ by $-\sigma(n,k)$. In the second series in (4.9), we can also replace the upper summation boundary by $n-1$. Numerical tests of the finite series evaluation (4.8)–(4.10) of integral $H_{-1}(n,p;b)$ in (4.2) are performed in Table IV by comparing with the Legendre asymptotics in Section 4.3 and to the Airy approximation (5.10) of Beltrami integrals.

4.2. Beltrami integrals with positive power-law exponent

Multiple b differentiation of identity (4.8) can be invoked to calculate the Beltrami integrals (4.1) at integer power-law index $\mu = m-1$, $m \geq 0$,

$$H_{m-1}(n,p;b) = \int_0^\infty k^{1+m} \mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k = \frac{(-1)^m}{2p^2}\frac{\mathrm{d}^m}{\mathrm{d}b^m}Q_n\left(1+\frac{b^2}{2p^2}\right) = \frac{(-1)^m}{2^{m+1}p^{m+2}}q_n^{(m)}\left(\frac{b}{2p}\right), \tag{4.11}$$

where $q_n(x)$ is the finite Legendre series (4.9), $q_n^{(m)}(x) = \mathrm{d}^m q_n(x)/\mathrm{d}x^m$, $q_n^{(0)}(x) = q_n(x)$, and $x = b/(2p)$.

For $m = 1$,

$$H_0(n,p;b) = \int_0^\infty k^2 \mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k = -\frac{1}{4p^3}q_n^{(1)}(b/(2p)), \tag{4.12}$$

where $q_n^{(1)}$ is the first derivative of $q_n(x)$ in (4.9),

$$\begin{aligned} q_n^{(1)}(x) = {} & -\frac{1}{x(1+x^2)} + (\log(1+x^2) - 2\log x)\sum_{k=1}^{n}\frac{[n,k]}{\Gamma(k)}x^{2k-1} \\ & - \frac{1}{1+x^2}\sum_{k=1}^{n}\frac{[n,k]}{k\Gamma(k)}x^{2k-1} + 2\sum_{k=1}^{n}\frac{[n,k]}{\Gamma(k)}(\psi(k+1)-\psi(n+1))x^{2k-1}. \end{aligned} \tag{4.13}$$

For $m = 2$, we obtain

$$H_1(n,p;b) = \int_0^\infty k^3 \mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k = \frac{1}{8p^4}q_n^{(2)}(b/(2p)), \tag{4.14}$$

where the second derivative of the Legendre series (4.9) reads

$$\begin{aligned} q_n^{(2)}(x) = {} & \frac{1+3x^2}{x^2(1+x^2)^2} + (\log(1+x^2) - 2\log x)\sum_{k=1}^{n}\frac{[n,k]}{\Gamma(k)}(2k-1)x^{2k-2} \\ & + \frac{2x^2}{(1+x^2)^2}\sum_{k=1}^{n}\frac{[n,k]}{k\Gamma(k)}x^{2k-2} - \frac{1}{1+x^2}\sum_{k=1}^{n}\frac{[n,k]}{k\Gamma(k)}(4k-1)x^{2k-2} \\ & + 2\sum_{k=1}^{n}\frac{[n,k]}{\Gamma(k)}(2k-1)(\psi(k+1)-\psi(n+1))x^{2k-2}. \end{aligned} \tag{4.15}$$

For $m = 3$, we find

$$H_2(n,p;b) = \int_0^\infty k^4 \mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k = -\frac{1}{16p^5}q_n^{(3)}(b/(2p)), \tag{4.16}$$

with the third derivative of $q_n(x)$in (4.9) given by

$$\begin{aligned} q_n^{(3)}(x) = {} & \frac{2(3x^2-1)}{x(1+x^2)^3}n(n+1) - 2\frac{1+3x^2+6x^4}{x^3(1+x^2)^3} \\ & - \frac{2}{1+x^2}\sum_{k=2}^{n}\frac{[n,k]}{k\Gamma(k)}(6k^2-6k+1)x^{2k-3} \\ & + \frac{6x^2}{(1+x^2)^2}\sum_{k=2}^{n}\frac{[n,k]}{k\Gamma(k)}(2k-1)x^{2k-3} + \frac{4x^2(1-x^2)}{(1+x^2)^3}\sum_{k=2}^{n}\frac{[n,k]}{k\Gamma(k)}x^{2k-3} \\ & + (\log(1+x^2) - 2\log x)\sum_{k=2}^{n}\frac{[n,k]}{\Gamma(k)}(2k-1)(2k-2)x^{2k-3} \\ & + 2\sum_{k=2}^{n}\frac{[n,k]}{\Gamma(k)}(2k-1)(2k-2)(\psi(k+1)-\psi(n+1))x^{2k-3}. \end{aligned} \tag{4.17}$$

The finite Legendre series $q_n^{(0,1,2,3)}(x)$ in (4.9), (4.13), (4.15), and (4.17) are suitable for high-precision calculations at low and moderate Bessel index n. Numerical tests of the series evaluation of integral H_0 in (4.12), H_1 in (4.14), and H_2 in (4.16) are given in Tables IV and V.

4.3. High-index Legendre asymptotics of Beltrami integrals

We derive a high-n approximation of the Beltrami integrals (4.11). The uniform asymptotic limit $n \to \infty$ of the Legendre function Q_n in (4.3) is [7, 8]

$$Q_n(\cosh x) \sim \frac{x^{1/2}}{\sinh^{1/2} x} K_0((n+1/2)x)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{4.18}$$

We put $\cosh x = \eta$, so that

$$x = \operatorname{arccosh}\eta = \operatorname{arcsinh}\sqrt{\eta^2 - 1} = \log\left(\eta + \sqrt{\eta^2 - 1}\right), \tag{4.19}$$

and

$$Q_n(\eta) \sim \frac{\operatorname{arccosh}^{1/2}\eta}{(\eta^2 - 1)^{1/4}} K_0((n+1/2)\operatorname{arccosh}\eta). \tag{4.20}$$

The asymptotic m-fold derivative $Q_n^{(m)} = \mathrm{d}^m Q_n/\mathrm{d}\eta^m$, $Q_n^{(0)} = Q_n$, reads, in leading order in n,

$$Q_n^{(m)}(\eta) \sim (-1)^m (n+1/2)^m \frac{\operatorname{arccosh}^{1/2}\eta}{(\eta^2 - 1)^{m/2+1/4}} K_m((n+1/2)\operatorname{arccosh}\eta), \tag{4.21}$$

which is obtained by repeated differentiation of (4.20), substituting $\operatorname{arccosh}'\eta = (\eta^2 - 1)^{-1/2}$. In leading order, we only need to differentiate the modified Bessel function in (4.20), $K_n'(z) = -K_{n+1}(z) + (n/z)K_n(z)$. Here, we can drop the second term and thus replace the mth derivative $K_0^{(m)}(z)$ by $(-1)^m K_m(z)$ to arrive at (4.21). We also note that $Q_n^m(\eta) = (\eta^2 - 1)^{m/2} Q_n^{(m)}(\eta)$ is an associated Legendre function of the second kind, of integer degree n and order m, with interval $[-1, 1]$ as branch cut [3].

Table V. Beltrami integrals $H_{1,2}(n,p;b)$, see (4.1), with $p = 1$, $b = 2.1 \times 10^{-4}$, and Bessel index n. The caption to Table IV applies. The finite Legendre series evaluation of $H_1(n,p;b)$ (column 2) is based on (4.14) and (4.15), and of $H_2(n,p;b)$ (column 5) on (4.16) and (4.17). The modified Bessel approximation (in columns 3 and 6) is based on (4.21) and (4.22), and the Airy approximation (in columns 4 and 7) on (5.10).

n	$H_1(n,p;b)$ Legendre	$H_1(n,p;b)$ modified Bessel	$H_1(n,p;b)$ Airy	$H_2(n,p;b)$ Legendre	$H_2(n,p;b)$ modified Bessel	$H_2(n,p;b)$ Airy
0	1.1337868605×10^7	1.1337868862×10^7	1.13378690×10^7	$1.0797969981 \times 10^{11}$	$1.0797969733 \times 10^{11}$	$1.07979700 \times 10^{11}$
1	1.1337872436×10^7	1.1337872633×10^7	1.13378727×10^7	$1.0797970219 \times 10^{11}$	$1.0797969971 \times 10^{11}$	$1.07979702 \times 10^{11}$
5	1.1337916442×10^7	1.1337916559×10^7	1.13379167×10^7	$1.0797973553 \times 10^{11}$	$1.0797973305 \times 10^{11}$	$1.07979735 \times 10^{11}$
10	1.1338026252×10^7	1.1338026329×10^7	1.13380264×10^7	$1.0797983076 \times 10^{11}$	$1.0797982828 \times 10^{11}$	$1.07979831 \times 10^{11}$
50	1.1340522250×10^7	1.1340522229×10^7	1.13405223×10^7	$1.0798273423 \times 10^{11}$	$1.0798273174 \times 10^{11}$	$1.07982734 \times 10^{11}$
100	1.1346642954×10^7	1.1346642890×10^7	1.13466430×10^7	$1.0799170599 \times 10^{11}$	$1.0799170350 \times 10^{11}$	$1.07991706 \times 10^{11}$
500	1.1455750718×10^7	1.1455750559×10^7	1.14557507×10^7	$1.0827101841 \times 10^{11}$	$1.0827101588 \times 10^{11}$	$1.08271018 \times 10^{11}$
1000	1.1642963251×10^7	1.1642963060×10^7	1.16429632×10^7	$1.0908809851 \times 10^{11}$	$1.0908809589 \times 10^{11}$	$1.09088098 \times 10^{11}$
2000	1.1949092882×10^7	1.1949092678×10^7	1.19490928×10^7	$1.1182687775 \times 10^{11}$	$1.1182687495 \times 10^{11}$	$1.11826877 \times 10^{11}$
10^4	7.9613281851×10^6	7.9613281108×10^6	7.96132819×10^6	$1.1320076538 \times 10^{11}$	$1.1320076320 \times 10^{11}$	$1.13200765 \times 10^{11}$
10^5	–	1.0810236001	1.08102357	–	1.1089216428×10^5	1.10892162×10^5

Returning to (4.11), we find

$$
\begin{aligned}
H_{-1}(n,p;b) &= \int_0^\infty k e^{-bk} j_n^2(pk)\mathrm{d}k = \frac{1}{2p^2} Q_n\left(1+\frac{b^2}{2p^2}\right),\\
H_0(n,p;b) &= \int_0^\infty k^2 e^{-bk} j_n^2(pk)\mathrm{d}k = -\frac{b}{2p^4} Q_n^{(1)}\left(1+\frac{b^2}{2p^2}\right),\\
H_1(n,p;b) &= \int_0^\infty k^3 e^{-bk} j_n^2(pk)\mathrm{d}k = \frac{1}{2p^4} Q_n^{(1)}\left(1+\frac{b^2}{2p^2}\right) + \frac{b^2}{2p^6} Q_n^{(2)}\left(1+\frac{b^2}{2p^2}\right),\\
H_2(n,p;b) &= \int_0^\infty k^4 e^{-bk} j_n^2(pk)\mathrm{d}k = -\frac{3b}{2p^6} Q_n^{(2)}\left(1+\frac{b^2}{2p^2}\right) - \frac{b^3}{2p^8} Q_n^{(3)}\left(1+\frac{b^2}{2p^2}\right).
\end{aligned}
\tag{4.22}
$$

The modified Bessel approximation is thus obtained by substituting the high-n limit of $Q_n^{(m)}$ as stated in (4.21), with $\eta = 1 + b^2/(2p^2)$. A numerical comparison of the modified Bessel approximation with the exact finite series evaluation of these integrals (in Sections 4.1 and 4.2) and their Airy approximation (5.10) is given in Tables IV and V.

5. Airy approximation of Weber and Beltrami integrals

We consider an arbitrary kernel function $g(k)$ and write

$$
D_{\mathrm{ssB}}(n,p;g) = \int_0^\infty g(k) j_n^2(kp) k^2 \mathrm{d}k. \tag{5.1}
$$

In the case of the Weber averages $E_\mu(n,p;a)$ in (2.1), we put $g(k) = k^\mu e^{-ak^2}$. Beltrami integrals $H_\mu(n,p;b)$ are defined by $g(k) = k^\mu e^{-bk}$, see (4.1). Otherwise, the density $g(k)$ need not be specified in this section.

We rescale the integration variable in (5.1) with $(n+1/2)/p$,

$$
D_{\mathrm{ssB}}(n,p;g) = \frac{(n+1/2)^3}{p^3} \int_0^\infty g\left(\frac{n+1/2}{p}x\right) j_n^2((n+1/2)x) x^2 \mathrm{d}x, \tag{5.2}
$$

and substitute the Nicholson approximation of the spherical Bessel function, valid for positive argument x and high Bessel index n [3],

$$
j_n((n+1/2)x) \sim \sqrt{\pi}\left(\frac{\xi(x)}{1-x^2}\right)^{1/4} \frac{\mathrm{Ai}((n+1/2)^{2/3}\xi(x))}{(n+1/2)^{5/6} x^{1/2}}. \tag{5.3}
$$

The variable $\xi(x)$ is defined by

$$
\begin{aligned}
\xi(x \ge 1) &= -\left(\frac{3}{2}\right)^{2/3}\left(\sqrt{x^2-1} - \arctan\sqrt{x^2-1}\right)^{2/3},\\
\xi(x \le 1) &= \left(\frac{3}{2}\right)^{2/3}\left(\operatorname{arctanh}\sqrt{1-x^2} - \sqrt{1-x^2}\right)^{2/3},
\end{aligned}
\tag{5.4}
$$

so that $\xi(x>1) < 0$ and $\xi(0<x<1) > 0$.

In this approximation, a squared Airy function appears in the integrand of (5.2), which admits the integral representation [9, 10]

$$
\mathrm{Ai}^2(z) = \frac{1}{2\pi^{3/2}} \int_0^\infty \cos\left(\frac{1}{12}t^3 + zt + \frac{\pi}{4}\right)\frac{\mathrm{d}t}{\sqrt{t}}, \tag{5.5}
$$

valid for real argument z. For large z, we can drop the cubic term in the argument of the cosine (Riemann–Lebesgue lemma):

$$
\mathrm{Ai}^2(z) \approx \frac{1}{2\pi^{3/2}} \int_0^\infty \cos\left(zt + \frac{\pi}{4}\right)\frac{\mathrm{d}t}{\sqrt{t}} = \frac{1}{2\pi}\frac{1}{(-z)^{1/2}}\theta(-z), \tag{5.6}
$$

where $\theta(z)$ is the Heaviside step function. Thus, we put $j_n^2((n+1/2)x) \approx 0$ in the interval $0<x<1$, where $\xi(x)$ is positive, which means to replace the lower integration boundary in integral (5.2) by 1. In the range $x \ge 1$, we introduce a new integration variable, $x = \sqrt{1+y}$, and substitute (5.6) into the squared Nicholson approximation (5.3), with $\xi\left(\sqrt{1+y}\right) = -2^{-2/3}y + O(y^2)$, to obtain

$$
\sqrt{1+y}\, j_n^2\left((n+1/2)\sqrt{1+y}\right) \approx \frac{1}{2(n+1/2)^2}\frac{1}{\sqrt{y}}, \tag{5.7}
$$

which covers the interval $0 < y < \infty$ at high n. Applying this approximation and the indicated variable change $y = x^2 - 1$ to integral (5.2), we obtain the Airy approximation of integral (5.1),

$$
D_{\mathrm{ssB}}(n,p;g) \approx \frac{n+1/2}{4p^3} \int_0^\infty g\left(\frac{n+1/2}{p}\sqrt{1+y}\right)\frac{\mathrm{d}y}{\sqrt{y}}. \tag{5.8}
$$

This is a steepest-descent approximation customarily employed in spectral fits of the cosmic microwave background (CMB) radiation, which can be derived in different ways by using Debye expansions [11] or Legendre asymptotics [12]. Its accuracy suffices to reproduce the multipole coefficients extracted from the CMB sky maps in the presently available resolution [1].

The Weber averages (2.1) with $g(k) = k^{\mu} e^{-ak^2}$ in (5.1) are thus approximated by

$$E_{\mu}(n,p;a) \approx \frac{(n+1/2)^{\mu+1}}{4p^{\mu+3}} \int_0^{\infty} (1+y)^{\mu/2} \exp\left[-\frac{a}{p^2}(n+1/2)^2(1+y)\right] \frac{dy}{\sqrt{y}}. \tag{5.9}$$

To connect this to the Debye expansion in Section 3, we note that $E_0(n,p;a)$ defined by (5.9) admits elementary integration, $4p^2E_0 \approx \sqrt{\pi/a}\exp[-(n+1/2)^2 a/p^2]$. This is just the leading order of the Debye expansion (3.14) and (3.15) in the limit of large $p^2/(an)$, where we can approximate $q \approx 1/y$ and $\eta - y \approx -1/(2y)$ in (3.17) and (3.18).

The Airy approximation of the Beltrami averages (4.1) (with $g(k) = k^{\mu} e^{-bk}$ in (5.1)) reads

$$H_{\mu}(n,p;b) \approx \frac{(n+1/2)^{\mu+1}}{4p^{\mu+3}} \int_0^{\infty} (1+y)^{\mu/2} \exp\left[-\frac{b}{p}(n+1/2)\sqrt{1+y}\right] \frac{dy}{\sqrt{y}}. \tag{5.10}$$

To connect to the Legendre asymptotics of the Beltrami integrals in (4.21) and (4.22), we perform a variable change $1 + y = (1+t)^2$ in integral (5.10) to find $2p^2H_{-1} \approx K_0((n+1/2)b/p)$. This coincides with the modified Bessel approximation of $H_{-1}(n,p;b)$ in the limit of small b/p, where we can put $\operatorname{arccosh}\eta \approx b/p$ and $\eta^2 - 1 \approx b^2/p^2$ in (4.21). The accuracy of the Airy approximations (5.9) and (5.10) is limited as exemplified in Tables I–VII, but they are numerically tame and also apply for noninteger exponents μ.

6. Beltrami integrals with negative power-law exponent

We calculate the integrals $H_{\mu}(n,p;b)$ in (4.1) for $\mu = -2, -3, -4, -5$. These integrals can be expressed as finite linear combinations of elementary functions, as is the case for positive exponents discussed in Section 4.2. Convergence conditions are stated after (4.1). We assume the parameter b in the exponential of (4.1) to be real and positive and invoke analytic continuation to complex b as final step if required. Negative integer power-law exponents $2+\mu$ (including zero) in integral (4.1) are obtained by repeated b integration of identity (4.8) over the interval $[0,b]$. The starting point is thus identity

$$\int_0^{\infty} k e^{-bk} j_n^2(pk) dk = \frac{1}{2p^2} q_n\left(\frac{b}{2p}\right), \tag{6.1}$$

where $q_n(x)$ is the Legendre series in (4.9) and $p > 0$, $b > 0$.

We will employ the Schafheitlin integrals [6]

$$H_{\mu-2}(n,p;0) = \int_0^{\infty} k^{\mu} j_n^2(pk) dk = \frac{2^{\mu-2}\pi}{p^{\mu+1}} \frac{\Gamma(1-\mu)}{\Gamma^2(1-\mu/2)} \frac{\Gamma(n+1+(\mu-1)/2)}{\Gamma(n+1-(\mu-1)/2)}, \tag{6.2}$$

convergent for $2n+2 > Re(1-\mu) > 0$, as well as the k-fold antiderivatives of the Legendre series $q_n(x)$ in (4.9),

$$q_n^{(-k)}(x) = \int_0^x \cdots \int_0^x q_n(x)(dx)^k, \tag{6.3}$$

which are explicitly calculated in Section 7 for $k = 1, 2, 3, 4$. These antiderivatives can be parameterized as in (4.8),

$$\int_0^b \cdots \int_0^b q_n\left(\frac{b}{2p}\right)(db)^k = (2p)^k q_n^{(-k)}\left(\frac{b}{2p}\right). \tag{6.4}$$

6.1. Integral $\int_0^{\infty} e^{-bk} j_n^2(pk) dk$

Integral $H_{-2}(n,p;b)$ in (4.1) is found by b integration of identity (6.1) over the interval $[0,b]$ using

$$\int_0^b e^{-bk} db = -\frac{1}{k}e^{-bk} + \frac{1}{k}, \tag{6.5}$$

which gives

$$H_{-2}(n,p;b) = \int_0^{\infty} e^{-bk} j_n^2(pk) dk = \int_0^{\infty} j_n^2(pk) dk - \frac{1}{2p^2}\int_0^b q_n\left(\frac{b}{2p}\right) db. \tag{6.6}$$

Here, we substitute, see (6.2),

$$\int_0^{\infty} j_n^2(pk) dk = \frac{\pi}{2p}\frac{1}{2n+1}, \tag{6.7}$$

which is convergent for $n \geq 0$, and use (6.4) with $k = 1$ to obtain

$$pH_{-2}(n,p;b) = \frac{\pi}{2}\frac{1}{2n+1} - q_n^{(-1)}\left(\frac{b}{2p}\right). \tag{6.8}$$

The antiderivative $q_n^{(-1)}(x)$ is defined in (6.4) and calculated in (7.2) as elementary finite series.

6.2. Integral $\int_0^\infty k^{-1}\mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k$

By integrating (6.1) twice with respect to b using

$$\int_0^b \int_0^b \mathrm{e}^{-bk}(\mathrm{d}b)^2 = \frac{1}{k^2}\mathrm{e}^{-bk} - \frac{1}{k^2} + \frac{b}{k}, \tag{6.9}$$

we obtain integral $H_{-3}(n,p;b)$ in (4.1) as

$$\begin{aligned} H_{-3}(n,p;b) &= \int_0^\infty \frac{1}{k}\mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k \\ &= \int_0^\infty \frac{1}{k} j_n^2(pk)\mathrm{d}k - b\int_0^\infty j_n^2(pk)\mathrm{d}k + \frac{1}{2p^2}\int_0^b \int_0^b q_n\left(\frac{b}{2p}\right)(\mathrm{d}b)^2. \end{aligned} \tag{6.10}$$

We substitute (6.7) and, see (6.2),

$$\int_0^\infty \frac{1}{k} j_n^2(pk)\mathrm{d}k = \frac{1}{2n(n+1)}, \tag{6.11}$$

which is convergent for $n \geq 1$, and use (6.4) with $k = 2$. Thus, we find for the scale-invariant case $\mu = -3$,

$$H_{-3}(n,p;b) = \frac{1}{2n(n+1)} - \frac{b}{2p}\frac{\pi}{2n+1} + 2q_n^{(-2)}\left(\frac{b}{2p}\right), \tag{6.12}$$

with the second antiderivative $q_n^{(-2)}(x)$ of the Legendre series (4.9) stated in (7.5).

6.3. Integral $\int_0^\infty k^{-2}\mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k$

To calculate integral $H_{-4}(n,p;b)$ in (4.1), we apply three-fold b integration to identity (6.1), which amounts to substituting

$$\int_0^b \int_0^b \int_0^b \mathrm{e}^{-bk}(\mathrm{d}b)^3 = -\frac{1}{k^3}\mathrm{e}^{-bk} + \frac{1}{k^3} - \frac{b}{k^2} + \frac{b^2}{2k}. \tag{6.13}$$

We arrive at

$$\begin{aligned} H_{-4}(n,p;b) &= \int_0^\infty \frac{1}{k^2}\mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k \\ &= \int_0^\infty \frac{1}{k^2} j_n^2(pk)\mathrm{d}k - b\int_0^\infty \frac{1}{k} j_n^2(pk)\mathrm{d}k \\ &\quad + \frac{b^2}{2}\int_0^\infty j_n^2(pk)\mathrm{d}k - \frac{1}{2p^2}\int_0^b \int_0^b \int_0^b q_n\left(\frac{b}{2p}\right)(\mathrm{d}b)^3. \end{aligned} \tag{6.14}$$

Here, we use the Schafheitlin integrals (6.7), (6.11), and

$$\int_0^\infty \frac{1}{k^2} j_n^2(pk)\mathrm{d}k = \frac{p\pi}{(2n+3)(2n+1)(2n-1)}, \tag{6.15}$$

the latter being convergent for $n \geq 1$. Finally, we substitute (6.4) (with $k = 3$) and find integral (4.1) with exponent $\mu = -4$ as

$$\begin{aligned} \frac{1}{p}H_{-4}(n,p;b) = &\frac{\pi}{(2n+3)(2n+1)(2n-1)} \\ &- \frac{b}{2p}\frac{1}{n(n+1)} + \frac{b^2}{4p^2}\frac{\pi}{2n+1} - 4q_n^{(-3)}\left(\frac{b}{2p}\right), \end{aligned} \tag{6.16}$$

where the third antiderivative $q_n^{(-3)}(x)$ of (4.9) is listed in (7.10).

6.4. Integral $\int_0^\infty k^{-3}\mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k$

Fourfold b integration of identity (6.1) by means of

$$\int_0^b \int_0^b \int_0^b \int_0^b \mathrm{e}^{-bk}(\mathrm{d}b)^4 = \frac{1}{k^4}\mathrm{e}^{-bk} - \frac{1}{k^4} + \frac{b}{k^3} - \frac{b^2}{2k^2} + \frac{b^3}{6k} \tag{6.17}$$

gives integral $H_{-5}(n,p;b)$ in (4.1) as

$$\begin{aligned} H_{-5}(n,p;b) &= \int_0^\infty \frac{1}{k^3} e^{-bk} j_n^2(pk)dk \\ &= \int_0^\infty \frac{1}{k^3} j_n^2(pk)dk - b\int_0^\infty \frac{1}{k^2} j_n^2(pk)dk + \frac{b^2}{2}\int_0^\infty \frac{1}{k} j_n^2(pk)dk \\ &\quad - \frac{b^3}{6}\int_0^\infty j_n^2(pk)dk + \frac{1}{2p^2}\int_0^b\int_0^b\int_0^b\int_0^b q_n\left(\frac{b}{2p}\right)(db)^4. \end{aligned} \tag{6.18}$$

Here, we substitute the integrals (6.7), (6.11), (6.15), and

$$\int_0^\infty \frac{1}{k^3} j_n^2(pk)dk = \frac{p^2}{3}\frac{1}{(n+2)(n+1)n(n-1)}, \tag{6.19}$$

the latter being convergent for $n \geq 2$, see (6.2). Finally, we employ (6.4) ($k = 4$) to obtain integral (4.1) with exponent $\mu = -5$ as

$$\begin{aligned} \frac{1}{p^2} H_{-5}(n,p;b) &= \frac{1}{3}\frac{1}{(n+2)(n+1)n(n-1)} \\ &\quad - \frac{b}{2p}\frac{2\pi}{(2n+3)(2n+1)(2n-1)} + \frac{b^2}{(2p)^2}\frac{1}{n(n+1)} \\ &\quad - \frac{1}{3}\frac{b^3}{(2p)^3}\frac{2\pi}{2n+1} + 8q_n^{(-4)}\left(\frac{b}{2p}\right). \end{aligned} \tag{6.20}$$

The antiderivative $q_n^{(-4)}(x)$ is composed of finite series and elementary functions and explicitly given in (7.16). Higher negative integer exponents μ can be dealt with in like manner by repeated b integration of (6.1). In particular, the multiple integrals $q_n^{(-k)}(x)$ in (6.3) stay elementary, see Appendix A. Positive integer power-law exponents $2+\mu$ in integral (4.1) admit elementary integration by repeated b differentiation of (6.1), see Section 4.2. Consistency checks of the identities (6.8), (6.12), (6.16), and (6.20) are also obtained by repeated b differentiation, employing the integral representation (4.1) and the differential version of (6.4),

$$\frac{d}{db} q_n^{(-k-1)}\left(\frac{b}{2p}\right) = \frac{1}{2p} q_n^{(-k)}\left(\frac{b}{2p}\right). \tag{6.21}$$

To complete the calculation of the Beltrami integrals in (6.8), (6.12), (6.16), and (6.20), we still need explicit finite series representations of the multiple antiderivatives $q_n^{(-k)}(x)$, stated in Section 7.

7. Antiderivatives of finite Legendre series: explicit formulae for high-precision evaluation of Beltrami integrals

We give explicit formulae for the multiple antiderivatives of the Legendre series $q_n(x) = Q_n(1+2x^2)$ in (4.9), which arise in the series evaluation of Beltrami integrals in Section 6. The antiderivatives are iteratively defined, see (6.3) and (6.4),

$$q_n^{(-k)}(x) = \int_0^x q_n^{(-k+1)}(t)dt = \int_0^x \cdots \int_0^x q_n(x)(dx)^k, \tag{7.1}$$

where k is a positive integer, n a nonnegative integer, and $q_n^{(0)}(x) = q_n(x)$, $dq_n^{(-k)}(x)/dx = q_n^{(-k+1)}(x)$. We will also use the Hankel symbol $[n,k]$ defined in (2.4), the rising factorial or Pochhammer symbol $(n)_k$ defined in (4.6), the psi function in (4.7), and the finite series $\sigma(n,k)$ in (4.10). In this section, we list $q_n^{(-k)}(x)$ for $k = 1,2,3,4$, which completes the calculation of the Beltrami integrals (6.8), (6.12), (6.16), and (6.20). In Appendix A, we sketch the systematic calculation of multiple antiderivatives $q_n^{(-k)}(x)$ as finite linear combinations of elementary functions. In this section, we merely state the results.

On integrating the Legendre series $q_n(x)$ in (4.9), we find

$$\begin{aligned} q_n^{(-1)}(x) &= \int_0^x q_n(t)dt \\ &= \left(\frac{1}{2}\log(1+x^2) - \log x\right)\sum_{k=0}^n \frac{[n,k]}{\Gamma(1+k)}\frac{x^{2k+1}}{2k+1} \\ &\quad + \sum_{k=0}^n \frac{[n,k]}{\Gamma(1+k)}\left(\frac{1}{2k+1} + \psi(k+1) - \psi(n+1)\right)\frac{x^{2k+1}}{(2k+1)_1} \\ &\quad - \sum_{k=0}^n a_1(k,n)\frac{(-1)^k}{2k+1}x^{2k+1} + S(n,1/2)\arctan x, \end{aligned} \tag{7.2}$$

where $S(n, 1/2)$ is defined in (A22) and calculated in (A26), and $a_1(k,n)$ denotes the series

$$a_1(k,n) = \sum_{j=k}^{n} \frac{[n,j]}{\Gamma(1+j)} \frac{(-1)^j}{(2j+1)_1}, \tag{7.3}$$

which satisfies the recursive relation

$$a_1(k+1,n) = a_1(k,n) - \frac{[n,k]}{\Gamma(1+k)} \frac{(-1)^k}{(2k+1)_1}. \tag{7.4}$$

$(n)_k$ is defined in (4.6), and $[n,k]$ in (2.4). We may also replace the difference $\psi(k+1) - \psi(n+1)$ in (7.2) by $-\sigma(n,k)$, see (4.10). The series $a_1(0,n)$ can be summed in closed form, $a_1(0,n) = S(n, 1/2) = 1/(2n+1)$, see (A26).

We turn to the second antiderivative $q_n^{(-2)}(x)$, obtained by integration of $q_n^{(-1)}(x)$ in (7.2),

$$\begin{aligned} q_n^{(-2)}(x) &= \int_0^x q_n^{(-1)}(t)\mathrm{d}t \\ &= \left(\frac{1}{2}\log(1+x^2) - \log x\right) \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \frac{x^{2k+2}}{(2k+1)_2} \\ &+ \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \psi(k+1) - \psi(n+1)\right) \frac{x^{2k+2}}{(2k+1)_2} \\ &- \sum_{k=0}^{n} a_2(k,n) \frac{(-1)^k}{2k+2} x^{2k+2} - \frac{1}{2} S(n,1) \log(1+x^2) + S(n, 1/2) x \arctan x. \end{aligned} \tag{7.5}$$

Here, we substitute $S(n, \alpha)$ as calculated in (A26) and (A29). The coefficients $a_2(k,n)$ are defined by the finite series

$$a_2(k,n) = \sum_{j=k}^{n} \frac{[n,j]}{\Gamma(1+j)} (-1)^j \left(\frac{1}{(2j+1)_1 (2k+1)_1} + \frac{1}{(2j+1)_2}\right), \tag{7.6}$$

which can also be written as

$$\begin{aligned} a_2(k,n) &= \frac{a_1(k,n)}{2k+1} + \tilde{a}_2(k,n), \\ \tilde{a}_2(k,n) &= \sum_{j=k}^{n} \frac{[n,j]}{\Gamma(1+j)} \frac{(-1)^j}{(2j+1)_2}, \end{aligned} \tag{7.7}$$

with series $a_1(k,n)$ in (7.3). Series $\tilde{a}_2(k,n)$ in (7.7) satisfies a recursive identity analogous to (7.4),

$$\tilde{a}_2(k+1,n) = \tilde{a}_2(k,n) - \frac{[n,k]}{\Gamma(1+k)} \frac{(-1)^k}{(2k+1)_2}. \tag{7.8}$$

We split series $\tilde{a}_2(k,n)$ into partial fractions,

$$\tilde{a}_2(k,n) = \sum_{j=k}^{n} \frac{[n,j]}{\Gamma(1+j)} (-1)^j \left(\frac{1}{2j+1} - \frac{1}{2j+2}\right), \tag{7.9}$$

so that $\tilde{a}_2(0,n)$ can be summed as $\tilde{a}_2(0,n) = S(n, 1/2) - S(n,1)$, see (A22), (A26), and (A29).

We turn to the third antiderivative $q_n^{(-3)}(x)$, obtained by integration of $q_n^{(-2)}(x)$ in (7.5),

$$\begin{aligned} q_n^{(-3)}(x) &= \int_0^x q_n^{(-2)}(t)\mathrm{d}t \\ &= \left(\frac{1}{2}\log(1+x^2) - \log x\right) \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \frac{x^{2k+3}}{(2k+1)_3} \\ &+ \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3} + \psi(k+1) - \psi(n+1)\right) \frac{x^{2k+3}}{(2k+1)_3} \\ &- \sum_{k=0}^{n} a_3(k,n) \frac{(-1)^k}{2k+3} x^{2k+3} - \frac{1}{2} S(n,1) x \log(1+x^2) \\ &+ \frac{1}{2} S(n, 3/2)(x - \arctan x) + \frac{1}{2} S(n, 1/2) x^2 \arctan x, \end{aligned} \tag{7.10}$$

where we substitute the coefficients $S(n,\alpha)$ listed in (A26) and (A29), as well as series

$$a_3(k,n)=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}(-1)^j\left(\frac{1}{(2j+1)_1(2k+1)_2}+\frac{1}{(2j+1)_2(2k+2)_1}+\frac{1}{(2j+1)_3}\right). \tag{7.11}$$

This series can be split as, see (7.7),

$$\begin{aligned}a_3(k,n)&=\frac{a_2(k,n)}{2k+2}+\tilde a_3(k,n),\\ \tilde a_3(k,n)&=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}\frac{(-1)^j}{(2j+1)_3},\end{aligned} \tag{7.12}$$

with $a_2(k,n)$ in (7.6). Series $\tilde a_3(k,n)$ satisfies the recursive identity, see (7.8),

$$\tilde a_3(k+1,n)=\tilde a_3(k,n)-\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{(2k+1)_3}, \tag{7.13}$$

and can be split into three partial fractions,

$$\tilde a_3(k,n)=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}(-1)^j\left(\frac{1}{2}\frac{1}{2j+1}-\frac{1}{2j+2}+\frac{1}{2}\frac{1}{2j+3}\right), \tag{7.14}$$

so that $\tilde a_3(0,n)$ can be summed as, see (A26) and (A29),

$$\tilde a_3(0,n)=\frac{1}{2}S(n,1/2)-S(n,1)+\frac{1}{2}S(n,3/2). \tag{7.15}$$

The fourth antiderivative $q_n^{(-4)}(x)$ is found by integrating $q_n^{(-3)}(x)$ in (7.10),

$$\begin{aligned}q_n^{(-4)}(x)&=\int_0^x q_n^{(-3)}(t)\mathrm{d}t\\ &=\left(\frac{1}{2}\log(1+x^2)-\log x\right)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{x^{2k+4}}{(2k+1)_4}\\ &\quad+\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\left(\frac{1}{2k+1}+\frac{1}{2k+2}+\frac{1}{2k+3}+\frac{1}{2k+4}\right.\\ &\qquad\left.+\psi(k+1)-\psi(n+1)\right)\frac{x^{2k+4}}{(2k+1)_4}\\ &\quad-\sum_{k=0}^{n}a_4(k,n)\frac{(-1)^k}{2k+4}x^{2k+4}-\frac{1}{4}S(n,1)x^2\log(1+x^2)\\ &\quad+\frac{1}{12}S(n,2)(\log(1+x^2)-x^2)+\frac{1}{2}S(n,3/2)x(x-\arctan x)\\ &\quad+\frac{1}{6}S(n,1/2)x^3\arctan x.\end{aligned} \tag{7.16}$$

Here, we substitute the respective constants $S(n,\alpha)$ in (A26) and (A29). The coefficients $a_4(k,n)$ are defined by the series

$$\begin{aligned}a_4(k,n)=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}(-1)^j&\left(\frac{1}{(2j+1)_1(2k+1)_3}+\frac{1}{(2j+1)_2}\frac{1}{(2k+2)_2}\right.\\ &\left.+\frac{1}{(2j+1)_3}\frac{1}{(2k+3)_1}+\frac{1}{(2j+1)_4}\right),\end{aligned} \tag{7.17}$$

which can be written as, see (7.7) and (7.12),

$$\begin{aligned}a_4(k,n)&=\frac{a_3(k,n)}{2k+3}+\tilde a_4(k,n),\\ \tilde a_4(k,n)&=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}\frac{(-1)^j}{(2j+1)_4},\end{aligned} \tag{7.18}$$

Table VI. Beltrami integrals $H_{-2,-3,-4}(n,p;b)$, see (4.1), with $p=1$, $b=2.1\times10^{-4}$. The caption to Table IV applies. The integrals H_{-3} and H_{-4} are convergent for Bessel indices $n\geq 1$. In columns 2, 4, and 6, the exact evaluation is given by means of finite Legendre series efficient at low and moderate Bessel index; the calculation of H_{-2} is explained in Section 6.1, of H_{-3} in Section 6.2, and of H_{-4} in Section 6.3. The Legendre evaluation is compared with the Airy approximation (5.10) listed in columns 3, 5, and 7.

n	$H_{-2}(n,p;b)$ Legendre	$H_{-2}(n,p;b)$ Airy	$H_{-3}(n,p;b)$ Legendre	$H_{-3}(n,p;b)$ Airy	$H_{-4}(n,p;b)$ Legendre	$H_{-4}(n,p;b)$ Airy
0	1.5697293640	1.56971719	–	–	–	–
1	5.2263681281 $\times10^{-1}$	5.22634994 $\times10^{-1}$	2.4989015077 $\times10^{-1}$	2.22112373 $\times10^{-1}$	2.0938702177 $\times10^{-1}$	1.16308628 $\times10^{-1}$
5	1.4197245320 $\times10^{-1}$	1.41972309 $\times10^{-1}$	1.6636771106 $\times10^{-2}$	1.64990300 $\times10^{-2}$	2.4375230748 $\times10^{-3}$	2.35685712 $\times10^{-3}$
10	7.4040403658 $\times10^{-2}$	7.40403640 $\times10^{-2}$	4.5298318339 $\times10^{-3}$	4.51952468 $\times10^{-3}$	3.4138039324 $\times10^{-4}$	3.38277493 $\times10^{-4}$
50	1.4957886768 $\times10^{-2}$	1.49578850 $\times10^{-2}$	1.9288035938 $\times10^{-4}$	1.92861138 $\times10^{-4}$	3.0095564420 $\times10^{-6}$	3.00836436 $\times10^{-6}$
100	7.2925982803 $\times10^{-3}$	7.29259784 $\times10^{-3}$	4.7924173895 $\times10^{-5}$	4.79229486 $\times10^{-5}$	3.7667711673 $\times10^{-7}$	3.76639067 $\times10^{-7}$
500	1.2151527774 $\times10^{-3}$	1.21515275 $\times10^{-3}$	1.7091430674 $\times10^{-6}$	1.70914107 $\times10^{-6}$	2.7443951534 $\times10^{-9}$	2.74438306 $\times10^{-9}$
1000	5.0285274976 $\times10^{-4}$	5.02852745 $\times10^{-4}$	3.6973176157 $\times10^{-7}$	3.69731637 $\times10^{-7}$	3.0181394516 $\times10^{-10}$	3.01813579 $\times10^{-10}$
2000	1.8076195146 $\times10^{-4}$	1.80761950 $\times10^{-4}$	7.0077519032 $\times10^{-8}$	7.00775113 $\times10^{-8}$	2.9287471019 $\times10^{-11}$	2.92874603 $\times10^{-11}$
10^4	4.3193036059 $\times10^{-6}$	4.31930357 $\times10^{-6}$	3.8154360596 $\times10^{-10}$	3.81543597 $\times10^{-10}$	3.4431690213 $\times10^{-14}$	3.44316885 $\times10^{-14}$
10^5	–	1.00804485 $\times10^{-15}$	–	9.86705247 $\times10^{-21}$	–	9.66622657 $\times10^{-26}$

Table VII. Beltrami integral $H_{-5}(n,p;b)$, see (4.1), with $p=1$, $b=2.1\times10^{-4}$. The caption to Table IV applies. This integral is convergent for Bessel index $n\geq 2$. The exact finite Legendre evaluation of H_{-5} in Section 6.4 is compared with the high-n Airy approximation (5.10).

n	$H_{-5}(n,p;b)$ Legendre	$H_{-5}(n,p;b)$ Airy
2	$1.3882607540\times10^{-2}$	8.5280572×10^{-3}
5	$3.9631314991\times10^{-4}$	3.6377853×10^{-4}
10	$2.7986571511\times10^{-5}$	2.7352277×10^{-5}
50	$5.0666289159\times10^{-8}$	5.0616252×10^{-8}
100	$3.1881303364\times10^{-9}$	3.1873294×10^{-9}
500	$4.6961147000\times10^{-12}$	4.6960642×10^{-12}
1000	$2.6027977479\times10^{-13}$	2.6027902×10^{-13}
2000	$1.2785389049\times10^{-14}$	1.27853784×10^{-14}
10^4	$3.1563618800\times10^{-18}$	3.15636161×10^{-18}
10^5	–	9.47681771×10^{-31}

where $a_3(k,n)$ is defined in (7.11). Series $\tilde{a}_4(k,n)$ in (7.18) satisfies a recursive relation analogous to (7.4), (7.8), and (7.13),

$$\tilde{a}_4(k+1,n)=\tilde{a}_4(k,n)-\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{(2k+1)_4}, \tag{7.19}$$

and can be written in four partial fractions,

$$\tilde{a}_4(k,n)=\sum_{j=k}^{n}\frac{[n,j]}{\Gamma(1+j)}(-1)^j\left(\frac{1}{6}\frac{1}{2j+1}-\frac{1}{2}\frac{1}{2j+2}+\frac{1}{2}\frac{1}{2j+3}-\frac{1}{6}\frac{1}{2j+4}\right). \tag{7.20}$$

Series $\tilde{a}_4(0, n)$ can be summed by means of (A22), (A26), and (A29) as

$$\tilde{a}_4(0, n) = \frac{1}{6}S(n, 1/2) - \frac{1}{2}S(n, 1) + \frac{1}{2}S(n, 3/2) - \frac{1}{6}S(n, 2). \tag{7.21}$$

A stringent consistency check of the antiderivatives (7.2), (7.5), (7.10), and (7.16) is provided by the identities $\mathrm{d}q_n^{(-k-1)}(x)/\mathrm{d}x = q_n^{(-k)}(x)$, $k = 1, 2, 3$, which can be checked by making use of the indicated properties of the series coefficients $a_i(k, n)$, see (7.7)–(7.9), (7.12)–(7.14), and (7.18)–(7.20).

The multiple integrals $q_n^{(-k)}(x)$, $k \geq 1$, defined in (7.1), are composed of three elementary functions, $\arctan x$, $\log x$, and $\log(1+x^2)$, as well as polynomials in x. The recursive compilation of the antiderivatives $q_n^{(-k)}(x)$ of the Legendre series (4.9) is explained in Appendix A. We use $q_n^{(-k)}(x)$ only in the open half-plane $Re x > 0$ because $x = b/(2p)$, $Re b > 0$, and $p > 0$, see (4.2) and (4.9). The definition of $\arctan x$ is $(\mathrm{i}/2)\log((1 - \mathrm{i}x)/(1 + \mathrm{i}x))$, and principal values are assumed for all logarithms. The explicit formulas for the antiderivatives $q_n^{(-k)}(x)$, $k = 1, \cdots, 4$, given in (7.2), (7.5), (7.10), and (7.16) complete the calculation of the Beltrami integrals $H_{-k-1}(n, p; b)$ in (6.8), (6.12), (6.16), and (6.20). Numerical checks of the antiderivatives $q_n^{(-k)}(x)$ are obtained by comparing the finite series representation of $H_{-k-1}(n, p; b)$, $k = 1, \cdots, 4$, in Section 6 with the Airy approximation (5.10), see Tables VI and VII.

8. Conclusion

We have developed techniques for the high-precision evaluation of Weber integrals $E_\mu = \int_0^\infty k^{2+\mu} \mathrm{e}^{-ak^2} j_n^2(pk)\mathrm{d}k$ and Beltrami integrals $H_\mu = \int_0^\infty k^{2+\mu}\, \mathrm{e}^{-bk} j_n^2(pk)\mathrm{d}k$, and tested them by comparison with high-index asymptotic expansions. These integrals arise in the multipole expansion of temperature fluctuations of the cosmic background radiation, where multipoles up to order $n \approx 10^4$ can be resolved. In the integrands, a and b are exponents with positive real part, and p is a positive scale parameter, which can be scaled out of the squared spherical Bessel function by rescaling the exponents, so that we use $p = 1$ in the tables. μ is an integer power-law exponent and $n \geq 0$ the integer Bessel index. The real parts of the exponents a and b are usually small in multipole expansions, see Tables I and IV, so that application of Laplace asymptotics is not an option.

The purpose of this article is (i) to find finite series representations of these integrals which allow safe evaluation at low and moderate Bessel index n in any desired precision; (ii) to obtain asymptotic approximations that can be used for moderate and large n; and (iii) to perform numerical tests by comparing the high-n asymptotics with the finite series expansions over a wide n range, see Tables I–VII. For instance, the Airy approximation of Weber and Beltrami integrals derived in Section 5 is of limited accuracy but turns out to be useful even for very low Bessel index.

Specifically, we derived explicit finite series representations of the Weber integrals E_μ for power-law exponents $\mu = 0, 2, 4$ and $\mu = -2, -4, -6$ in Section 2. The high-n Debye approximation for $\mu = 0, 2, 4$ is derived in Section 3 and the Airy asymptotics for real μ in Section 5. Numerical tests are described in the captions to Tables I–III.

Finite series expansions of Beltrami integrals H_μ with integer exponents $\mu = -1, 0, 1, 2$ are given in Section 4. The Legendre asymptotics of these integrals is discussed in Section 4.3, and their Airy approximation in Section 5; numerical examples covering a wide range of Bessel indices are given in Tables IV and V.

In Sections 6 and 7, we obtained explicit finite series representations of Beltrami integrals with power-law exponents $\mu = -2, -3, -4, -5$. These series expansions are compared with the Airy approximation in Tables VI and VII. In this case, for integer exponents $\mu \leq -2$, a finite series evaluation of H_μ requires multiple antiderivatives of Legendre series, whose iterative calculation is explained in Appendix A.

APPENDIX A. Multiple integrals of finite Legendre series: outline of iterative calculation

We outline the systematic calculation of multiple antiderivatives $q_n^{(-k)}(x)$ of the Legendre series $q_n(x) = Q_n(1+2x^2)$, see (4.3) and (4.9), as defined in (7.1). In particular, we derive the antiderivatives $q_n^{(-k)}(x)$ for $k = 1, 2, 3, 4$, which are stated in (7.2), (7.5), (7.10), and (7.16).

We start with some prerequisites relating to the geometric series

$$\sum_{j=0}^{k} (-1)^j x^{2j} = \frac{(-1)^k x^{2(k+1)} + 1}{x^2 + 1} =: g_k(x), \tag{A1}$$

and its n-fold antiderivative

$$A_{k,0,n}(x) := \sum_{j=0}^{k} \frac{(-1)^j x^{2j+n}}{(2j+1)_n} = \int_0^x \cdots \int_0^x g_k(x)(\mathrm{d}x)^n, \tag{A2}$$

where we use the rising factorial $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. More generally, we will need the antiderivatives of $g_k(x)x^l$,

$$A_{k,l,n}(x) := \sum_{j=0}^{k} \frac{(-1)^j x^{2j+l+n}}{(2j+1+l)_n} = \int_0^x \cdots \int_0^x g_k(x)x^l(\mathrm{d}x)^n, \tag{A3}$$

where k, l and n are nonnegative integers, and $A_{k,l,0}(x) = g_k(x)x^l$. We also note the identities

$$\int_0^x A_{k,l,n}(x)\mathrm{d}x = A_{k,l,n+1}(x), \quad A_{k,l,n}(x) = \frac{\mathrm{d}}{\mathrm{d}x}A_{k,l,n+1}(x),$$
$$A_{k+1,l,n}(x) = \frac{x^{l+n}}{(1+l)_n} - A_{k,l+2,n}(x). \tag{A4}$$

To assemble the antiderivatives $q_n^{(-k)}(x)$ of the Legendre series (4.9), we need some integrals that can be expressed in terms of the finite series $A_{k,l,n}(x)$ in (A3). First,

$$\begin{aligned} I_1(x,k) &= \int_0^x t^{2k}\log(1+t^2)\mathrm{d}t \\ &= -\frac{2(-1)^k}{2k+1}A_{k,0,1}(x) + \frac{x^{2k+1}}{2k+1}\log(1+x^2) + \frac{2(-1)^k}{2k+1}\arctan x, \end{aligned} \tag{A5}$$

so that $I_1'(x,k) = x^{2k}\log(1+x^2) =: I_0(x,k)$. Odd powers can be dealt with in like manner,

$$\int_0^x t^{2k+1}\log(1+t^2)\mathrm{d}t = -\frac{2(-1)^k}{2k+2}A_{k,1,1}(x) + \frac{x^{2k+2}+(-1)^k}{2k+2}\log(1+x^2). \tag{A6}$$

The integrals (A6) are needed to calculate the antiderivative of $I_1(x,k)$ in (A5),

$$\begin{aligned} I_2(x,k) &= \int_0^x\int_0^x t^{2k}\log(1+t^2)(\mathrm{d}t)^2 \\ &= -\frac{2(-1)^k}{2k+1}A_{k,0,2}(x) - \frac{2(-1)^k}{(2k+1)_2}A_{k,1,1}(x) \\ &\quad + \frac{2(-1)^k}{2k+1}x\arctan x - \frac{(-1)^k}{2k+2}\log(1+x^2) + \frac{x^{2k+2}\log(1+x^2)}{(2k+1)_2}. \end{aligned} \tag{A7}$$

The third antiderivative of $I_0(x,k)$ reads

$$\begin{aligned} I_3(x,k) &= \int_0^x\int_0^x\int_0^x t^{2k}\log(1+t^2)(\mathrm{d}t)^3 \\ &= -\frac{2(-1)^k}{2k+1}A_{k,0,3}(x) - \frac{2(-1)^k}{(2k+1)_2}A_{k,1,2}(x) - \frac{2(-1)^k}{(2k+1)_3}A_{k,2,1}(x) \\ &\quad + \frac{(-1)^k}{2k+1}x^2\arctan x - \frac{(-1)^k}{2k+2}x\log(1+x^2) \\ &\quad + 2(-1)^k\left(\frac{1}{2k+2} - \frac{1}{2}\frac{1}{2k+1} + \frac{1}{(2k+1)_3}\right)(x-\arctan x) \\ &\quad + \frac{1}{(2k+1)_3}x^{2k+3}\log(1+x^2), \end{aligned} \tag{A8}$$

where we made use of identity $A_{k+1,0,1}(x) = x - A_{k,2,1}(x)$ in (A4).

The antiderivative of $I_3(x,k)$ is

$$\begin{aligned} I_4(x,k) &= \int_0^x\int_0^x\int_0^x\int_0^x t^{2k}\log(1+t^2)(\mathrm{d}t)^4 \\ &= -\frac{2(-1)^k}{2k+1}A_{k,0,4}(x) - \frac{2(-1)^k}{(2k+1)_2}A_{k,1,3}(x) \\ &\quad - \frac{2(-1)^k}{(2k+1)_3}A_{k,2,2}(x) - \frac{2(-1)^k}{(2k+1)_4}A_{k,3,1}(x) \\ &\quad + \frac{1}{3}\frac{(-1)^k}{2k+1}x^3\arctan x - \frac{1}{2}\frac{(-1)^k}{2k+2}x^2\log(1+x^2) \\ &\quad - 2(-1)^k\left(\frac{1}{2k+2} - \frac{1}{2}\frac{1}{2k+1} + \frac{1}{(2k+1)_3}\right)x\arctan x \\ &\quad + \frac{1}{(2k+1)_4}x^{2k+4}\log(1+x^2) \\ &\quad + (-1)^k x^2\left(\frac{3}{2}\frac{1}{2k+2} - \frac{2}{3}\frac{1}{2k+1} + \frac{1}{(2k+1)_3} + \frac{1}{(2k+1)_4}\right) \\ &\quad + (-1)^k\log(1+x^2)\left(\frac{1}{2}\frac{1}{2k+2} - \frac{1}{3}\frac{1}{2k+1} + \frac{1}{(2k+1)_3} - \frac{1}{(2k+1)_4}\right), \end{aligned} \tag{A9}$$

where we employed identity $A_{k+1,1,1}(x) = x^2/2 - A_{k,3,1}(x)$, see (A4). In these iterative calculations, $I_n(x,k) = \int_0^x I_{n-1}(t,k)\mathrm{d}t$, we used the elementary integrals

$$t_n(x) = \int_0^x t^n \arctan t\mathrm{d}t, \quad t_0(x) = x\arctan x - \frac{1}{2}\log(1+x^2),$$
$$t_1(x) = \frac{1}{2}(1+x^2)\arctan x - \frac{x}{2}, \quad t_2(x) = \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{1}{6}\log(1+x^2). \tag{A10}$$

Apparently, $I'_{n+1}(x,k) = I_n(x,k)$, which serves as a consistency check of the antiderivatives (A5) and (A7)–(A9). When differentiating, we employ the differential identity in (A4) and make use of the fact that series $A_{k,l,0}(x)$ can be summed in closed form as $g_k(x)x^l$, with $g_k(x)$ in (A1).

In addition to the integrals $I_n(x,k)$, we need multiple antiderivatives of $L_0(x,k) = x^{2k}\log x$, that is,

$$L_n(x,k) = \int_0^x L_{n-1}(t,k)\mathrm{d}t$$
$$= \frac{x^{2k+n}}{(2k+1)_n}\log x - \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{2k+n}\right)\frac{x^{2k+n}}{(2k+1)_n}, \tag{A11}$$

with $n \geq 1$, as well as the antiderivatives of $P_0(x,k) = x^{2k}$,

$$P_n(x,k) = \int_0^x P_{n-1}(t,k)\mathrm{d}t = \frac{x^{2k+n}}{(2k+1)_n}. \tag{A12}$$

The i-fold antiderivatives $q_n^{(-i)}(x)$ of the Legendre series $q_n(x)$, listed in Section 7 for $i = 1,2,3,4$, are assembled from the integrals $I_n(x,k)$, $L_n(x,k)$, and $P_n(x,k)$, calculated in (A5)–(A12). Substituting the Legendre series (4.9) into the multiple integral (7.1), we find

$$q_n^{(-i)}(x) = \frac{1}{2}A_i(x,n) - B_i(x,n) + C_i(x,n), \tag{A13}$$

where

$$A_i(x,n) = \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} I_i(x,k), \tag{A14}$$

$$B_i(x,n) = \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} L_i(x,k), \tag{A15}$$

$$C_i(x,n) = \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} (\psi(k+1) - \psi(n+1))P_i(x,k). \tag{A16}$$

In series $A_i(x,n)$, we interchange the k summation with the j summation of $A_{k,l,n}(x)$ (defined in (A3) and occurring in the series representation of the integrals $I_i(x,k)$),

$$\sum_{k=0}^{n}\sum_{j=0}^{k} = \sum_{j=0}^{n}\sum_{k=j}^{n}. \tag{A17}$$

In the following, we list $A_i(x,n)$ for $i = 1,2,3,4$. By making use of the antiderivative $I_1(x,k)$ in (A5) and the summation interchange (A17), we find, see (A14),

$$A_1(x,n) = \log(1+x^2)\sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \frac{x^{2k+1}}{2k+1}$$
$$- 2\sum_{j=0}^{n} a_1(j,n)\frac{(-1)^j}{2j+1}x^{2j+1} + 2\arctan x \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)} \frac{(-1)^k}{2k+1}, \tag{A18}$$

where the finite series $a_1(j,n)$ is stated in (7.3). The antiderivative $q_n^{(-1)}(x)$ recorded in (7.2) is obtained by means of (A13) with (A15), (A16), and (A18) substituted.

As for the second antiderivative $q_n^{(-2)}(x)$, see (A13), we obtain $A_2(x,n)$ in (A14) as

$$A_2(x,n) = \log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{x^{2k+2}}{(2k+1)_2} - 2\sum_{j=0}^{n} a_2(j,n)\frac{(-1)^j}{2j+2}x^{2j+2} - \log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+2} + 2x\arctan x\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+1}, \tag{A19}$$

with series $a_2(j,n)$ defined in (7.6). Here, we used $I_2(x,k)$ in (A7) and interchanged summations according to the prescription (A17). The final result for $q_n^{(-2)}(x)$ is stated in (7.5).

Regarding the calculation of $q_n^{(-3)}(x)$ in (A.13), we can proceed analogously. Series $A_3(x,n)$ in (A14) is found by substituting integral $I_3(x,k)$, see (A8), and by performing the interchange (A17),

$$\begin{aligned} A_3(x,n) = {} & \log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{x^{2k+3}}{(2k+1)_3} \\ & - 2\sum_{j=0}^{n} a_3(j,n)\frac{(-1)^j}{2j+3}x^{2j+3} - x\log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+2} \\ & + 2(x-\arctan x)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}(-1)^k\left(\frac{1}{(2k+1)_3}+\frac{1}{2k+2}-\frac{1}{2}\frac{1}{2k+1}\right) \\ & + x^2\arctan x\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+1}. \end{aligned} \tag{A20}$$

Series $a_3(j,n)$ is defined in (7.11), and the final result for $q_n^{(-3)}(x)$ is recorded in (7.10).

Finally, we substitute integral $I_4(x,n)$ (calculated in (A9)) into series $A_4(x,n)$ (defined in (A14)), and interchange summations as in (A17), to obtain

$$\begin{aligned} A_4(x,n) = {} & \log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{1}{(2k+1)_4}x^{2k+4} \\ & - 2\sum_{j=0}^{n} a_4(j,n)\frac{(-1)^j}{2j+4}x^{2j+4} - \frac{1}{2}x^2\log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+2} \\ & + \log(1+x^2)\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}(-1)^k\left(\frac{1}{2}\frac{1}{2k+2}-\frac{1}{3}\frac{1}{2k+1}+\frac{1}{(2k+1)_3}-\frac{1}{(2k+1)_4}\right) \\ & + x^2\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}(-1)^k\left(\frac{3}{2}\frac{1}{2k+2}-\frac{2}{3}\frac{1}{2k+1}+\frac{1}{(2k+1)_3}+\frac{1}{(2k+1)_4}\right) \\ & - 2x\arctan x\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}(-1)^k\left(\frac{1}{2k+2}-\frac{1}{2}\frac{1}{2k+1}+\frac{1}{(2k+1)_3}\right) \\ & + \frac{1}{3}x^3\arctan x\sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2k+1}. \end{aligned} \tag{A21}$$

Series $a_4(j,n)$ is defined in (7.17). The fourth antiderivative $q_n^{(-4)}(x)$ of the Legendre series (4.9) is assembled via (A13) by substitution of (A15), (A16), and (A21). The final result $q_n^{(-4)}(x)$ is stated in (7.16).

In (A18)–(A21), we encounter series of type

$$S(n,\alpha) = \sum_{k=0}^{n}\frac{[n,k]}{\Gamma(1+k)}\frac{(-1)^k}{2(k+\alpha)}, \tag{A22}$$

where $Re\alpha > 0$, and the Hankel symbol $[n,k]$ is defined in (2.4). All finite series arising in (A18)–(A21), which do not involve powers of x, can be reduced to $S(n,\alpha)$ by partial fraction decomposition. Series (A22) can be traced back to the Legendre polynomial $P_n(x)$ in representation (4.4), by performing a variable change $x = 1 - 2y^2$,

$$P_n(1-2y^2) = \sum_{k=0}^{n} \frac{[n,k]}{\Gamma(1+k)}(-1)^k y^{2k} = {}_2F_1(-n, n+1; 1; y^2). \tag{A23}$$

Thus, we may write series (A22) as

$$S(n,\alpha) = \int_0^1 P_n(1-2y^2) y^{2\alpha-1} \mathrm{d}y = \frac{1}{2^{\alpha+1}} \int_0^1 P_n(x)(1-x)^{\alpha-1} \mathrm{d}x. \tag{A24}$$

By making use of (A23) and a standard integral of the hypergeometric function [13], we find

$$S(n,\alpha) = \frac{1}{2} \int_0^1 {}_2F_1(-n, n+1; 1; x) x^{\alpha-1} \mathrm{d}x = \frac{1}{2} \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1+\alpha)}, \tag{A25}$$

where $n \geq 0$ is a nonnegative integer, and $Re\alpha > 0$. In this way, we have summed series (A22). For instance,

$$S(n, 1/2) = \frac{1}{2n+1}, \quad S(n, 3/2) = \frac{-1}{(2n+3)(2n+1)(2n-1)}. \tag{A26}$$

Singularities emerge in the gamma functions in (A25) at positive integer α. In this case, epsilon expansion is needed. We put $\alpha = m + \varepsilon$, $m \geq 1$, and use the expansion

$$\Gamma(-j+\varepsilon) = \frac{(-1)^j}{\Gamma(j+1)} \left(\frac{1}{\varepsilon} + \psi(j+1) + \mathrm{O}(\varepsilon) \right) \tag{A27}$$

valid for integer $j \geq 0$, where the psi function is defined in (4.7). We find $S(n, m+\varepsilon) = \mathrm{O}(\varepsilon)$ for $n \geq m$ and

$$S(n, m+\varepsilon) = \frac{1}{2} \frac{(-1)^n \Gamma^2(m)}{\Gamma(1+m+n)\Gamma(m-n)} + \mathrm{O}(\varepsilon) \tag{A28}$$

valid for $n < m$. Thus,

$$\begin{aligned} &S(n=0,1) = 1/2, \quad S(n \geq 1, 1) = 0, \\ &S(n=0,2) = 1/4, \quad S(n=1,2) = -1/12, \quad S(n \geq 2, 2) = 0. \end{aligned} \tag{A29}$$

When assembling the antiderivatives $q_n^{(-i)}(x)$ in Section 7, we make extensive use of series $S(n,\alpha)$ in (A22) for half-integer and integer α as listed in (A26) and (A29).

Finally we note a consistency check of the antiderivatives $q_n^{(-i)}(x)$ in (7.2), (7.5), (7.10), and (7.16). The numerical calculation of $q_n^{(-i)}(x)$ can directly be based on (A13)–(A16), without the interchange of indices (A17) in $A_i(x,n)$. That is, the coefficients $A_i(x,n)$ required in (A13) are calculated by substituting the integrals $I_i(x,k)$ into series (A14) and by performing the k summation as indicated in (A14). Closed analytic expressions of the integrals $I_i(x,k)$ are given in (A5) and (A7)–(A9), where we use the finite series representation (A3) of the antiderivatives $A_{k,l,n}(x)$ in the numerical evaluation.

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