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Tachyonic synchrotron radiation

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Abstract

The emission of superluminal quanta (tachyons) by ultra-relativistic electrons gyrating in magnetic fields is investigated. The tachyonic Liénard–Wiechert potentials of helically orbiting charges are derived, as well as the superluminal energy flux and the transversal and longitudinal spectral densities. We calculate the tachyonic synchrotron power, its angular dependence, the mean energy of the superluminal quanta radiated, tachyonic emission rates, the spectral maxima, critical and break frequencies, and we identify the Stokes parameter of the longitudinal radiation. The tachyonic energy densities are averaged with electronic power-law distributions, and the spectral indices are determined. Quantitative estimates are given for superluminal synchrotron radiation generated in storage rings, the Jovian magnetosphere, and supernova remnants. The spectral density of Jupiter's tachyonic X-ray emission is inferred from radio fluxes obtained from SL9 pre-impact observations and the Cassini fly-by, and we identify the tachyonic spectral peak at 2 keV in the ROSAT and Einstein spectral maps. We scrutinize multiwavelength observations of galactic supernova remnants, pointing out evidence in their wideband spectra for the TeV γ-radiation to be tachyonic rather than a consequence of inverse Compton scattering or pion decay. In the Crab Nebula, the electronic source population generating this radiation extends beyond the 'knee' of the cosmic ray spectrum.

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1. Introduction

We will study the spectral densities of superluminal radiation fields generated by ultra-relativistic particles in helical and circular orbits (tachyonic synchrotron radiation). We will discuss examples with orbital radii varying over some 15 orders of magnitude, from storage rings via planetary radiation belts to shell supernova remnants and plerions. Superluminal quanta are a sort of photons with negative mass-square, a quantized Proca field minimally coupled to subluminal matter [1–4]. The coupling strength of tachyons to matter is determined by the tachyonic fine structure constant; the ratio of electric and tachyonic fine structure constants is estimated from Lamb shifts in hydrogenic ions, $\alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$, like the tachyon-electron mass ratio, $m_t/m_e \approx 1/238$, cf. Ref. [5].

Although the Proca equation with negative mass-square is formally close to electromagnetic theory, there are some marked differences such as the third degree of freedom, longitudinally polarized quanta. In the ultra-relativistic limit studied here, the longitudinal radiation is even more pronounced than its transversal counterpart. If tachyons are radiated in the MeV range or above, the tachyon mass is negligible in the dispersion relation, so that they propagate very nearly at the speed of light, and then the longitudinal polarization gets crucial in distinguishing them from photons. At low energies, from soft X-rays down to radio frequencies, the tachyon mass dominates, so that tachyons cannot exceed a wavelength of roughly 6 Å. Low-energy tachyons can thus be discerned from photons by their wavelength, polarization, and speed. A basic difference to Maxwell's theory is the residual radiation in the limit of infinite curvature radius. Particles in linear uniform motion with regard to the cosmic microwave background can radiate superluminal quanta. In the ultra-relativistic limit, the spectrum of this radiation is not sharply peaked, extending over a broad frequency range, from the spectral maximum determined by the tachyon mass up to a break frequency depending on the Lorentz factor of the inertial source [6]. We will consider a finite gyroradius, so that the frequencies radiated extend beyond this break frequency by virtue of curvature radiation, which generates a smooth exponential cutoff in the tachyonic spectral densities.

The existence of superluminal quanta has not been established so far; Lamb shifts in hydrogen-like ions, hyperfine splittings, radiative transitions in Rydberg atoms, and ionization cross-sections provide circumstantial evidence [5]. Here, we will search the Jovian magnetosphere and galactic synchrotron nebulae. As for Jupiter's aurora, thanks to the SL9 comet impact and the Cassini spacecraft encounter, the synchrotron radio flux has been measured at various λ, from 90 to 2 cm, and can be used to piece together the electron density. This in turn allows us to infer the tachyon flux emitted in hard X-rays, as well as the slope and the break frequencies of the high-energy tail of the tachyonic spectral density. The ROSAT and Einstein spectral maps indicate line spectra of highly stripped heavy ions in soft X-rays, but also a third radiation peak at 2 keV, well separated from the line spectra, which will be shown to coincide with the peak of the tachyonic spectral density.

In search of high-energy tachyon radiation, we will study galactic supernova remnants with TeV γ-ray spectra, that is, the Crab, Cas A, the remnant of SN 1006, and RX

J1713.7–3946, the latter in connection with the nearby GeV γ-ray source 3EG J1714–3857. We will analyze their broadband spectra, from soft X-rays upwards, the spectral breaks and slopes. We will conclude that the EGRET flux data and, in the case of the Crab, the COMPTEL data in the low MeV region, support the identification of their MeV to TeV γ-ray spectra as tachyonic. We will determine the range and index of the power-law electron density generating the superluminal γ-radiation, and conclude that it is distinct from the electron population emitting the electromagnetic synchrotron radiation in the radio-to-X-ray bands.

In Section 2, we derive the tachyonic flux vectors and the integral energy flux emitted by helically moving charges, the formalism of tachyonic synchrotron and cyclotron radiation, that is. In Section 3, we specialize to charges in circular, ultra-relativistic motion (subluminal, with high Lorentz factors). We perform the asymptotic summation of the multipole expansion of the superluminal radiation field and calculate the transversal and longitudinal spectral densities. In Section 4, we integrate these densities to obtain the power radiated, the tachyonic number counts (emission rates), mean energies, and polarization ratios. We average the superluminal spectral densities with electronic power-law distributions, and relate the power-law index to the tachyonic spectral index. In Section 5, we first give numerical estimates for tachyon radiation in storage rings and compare with electromagnetic synchrotron radiation. We then turn to the Jovian magnetosphere, and infer the slopes of the tachyonic spectral density from the radio electrons. Jupiter's tachyon spectrum extends over the hard and high-energy X-ray bands, and we identify the tachyonic spectral peak in the ROSAT and Einstein spectral maps. In Section 6, we scrutinize galactic supernova remnants with known TeV γ-ray fluxes, and identify tachyonic spectral slopes in their multi-band spectra. The electron density generating the superluminal γ-rays is inferred from the tachyonic break energies and spectral slopes. The conclusions are stated at the end of Sections 5 and 6. In Appendix A, we sketch the stationary phase asymptotics of the tachyonic spectral densities and the power asymptotics.

2. Superluminal radiation by helically moving charges

The superluminal radiation field solves the Proca equation with negative mass-square, $(\Box + m_t^2)A_\alpha = -c^{-1}j_\alpha$, subjected to the Lorentz condition $A^\mu_{,\mu} = 0$. The sign conventions for metric and d'Alembertian are $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$, respectively. The tachyon mass has the dimension of an inverse length, a shortcut for $m_t c/\hbar$, estimated as $m_t/m_e \approx 1/238$ from Lamb shifts in hydrogenic systems [5]. We will mainly work in Fourier space, representing the spatial component of the vector potential as

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\mathbf{A}}(\mathbf{x},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \quad \hat{\mathbf{A}}^*(\mathbf{x},\omega) = \hat{\mathbf{A}}(\mathbf{x},-\omega)\,, \tag{2.1}$$

and analogous relations hold for the time component and the current density. Fourier transforms will be defined as in (2.1) and denoted by hats.

In Eqs. (2.2)–(2.10), we summarize the classical radiation theory, tachyon radiation by arbitrarily moving charges [4,6]. The Fourier transform of the current is split into a transversal and longitudinal component,

$$\hat{\mathbf{j}}^{\mathrm{T}}(\mathbf{x}',\mathbf{x},\omega) := \hat{\mathbf{j}}(\mathbf{x}',\omega) - \mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{j}}(\mathbf{x}',\omega)), \quad \hat{\mathbf{j}}^{\mathrm{L}}(\mathbf{x}',\mathbf{x},\omega) := \mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{j}}(\mathbf{x}',\omega)) , \tag{2.2}$$

where $\mathbf{n}=\mathbf{x}/r$, so that the respective components of the asymptotic radiation field read,

$$\hat{\mathbf{A}}^{\mathrm{T,L}}(\mathbf{x},\omega) \sim \frac{1}{4\pi c r}\exp(\mathrm{i}k(\omega)r)\hat{\mathbf{J}}^{\mathrm{T,L}}(\mathbf{x},\omega),$$

$$k(\omega) := \mathrm{sign}(\omega)\sqrt{\omega^2/c^2 + m_{\mathrm{t}}^2} , \tag{2.3}$$

$$\hat{\mathbf{J}}^{\mathrm{T,L}}(\mathbf{x},\omega) := \int \mathrm{d}\mathbf{x}'\hat{\mathbf{j}}^{\mathrm{T,L}}(\mathbf{x}',\mathbf{x},\omega)\exp(-\mathrm{i}k(\omega)\mathbf{n}\cdot\mathbf{x}') . \tag{2.4}$$

We consider the trajectory of a subluminal particle $\mathbf{x}_0(t)$, $\mathbf{v}=\dot{\mathbf{x}}_0$, carrying tachyonic charge q, so that the current density admits the Fourier transform

$$\hat{\mathbf{j}}(\mathbf{x},\omega) = q\int_{-\infty}^{+\infty} \mathbf{v}(t)\delta(\mathbf{x}-\mathbf{x}_0(t))\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t . \tag{2.5}$$

The transversal and longitudinal components of the velocity are $\mathbf{v}^{\mathrm{T}}(\mathbf{x},t) := \mathbf{v} - \mathbf{n}(\mathbf{n}\cdot\mathbf{v})$ and $\mathbf{v}^{\mathrm{L}}(\mathbf{x},t) := \mathbf{n}(\mathbf{n}\cdot\mathbf{v})$, respectively, cf. (2.2), so that transform (2.4) of the current can be assembled as

$$\hat{\mathbf{J}}^{\mathrm{T,L}}(\mathbf{x},\omega) = q\int_{-\infty}^{+\infty} \mathrm{d}t\mathbf{v}^{\mathrm{T,L}}(\mathbf{x},t)\exp[\mathrm{i}(\omega t - k(\omega)\mathbf{n}\cdot\mathbf{x}_0(t))] . \tag{2.6}$$

The asymptotic Liénard–Wiechert potentials of the tachyonic charge are thus given by (2.3) with this $\hat{\mathbf{J}}^{\mathrm{T,L}}$ inserted. Once these potentials are known, the time-averaged energy flux can be derived by a standard procedure [6]. We find the transversal flux,

$$\langle \mathbf{S}^{\mathrm{T}}\rangle \sim \frac{\mathbf{n}}{4(2\pi)^4c^2r^2}\frac{2\pi}{T}\int\int_{-\infty}^{+\infty} \omega k(\omega)\delta_{(1)}(\omega-\omega';T)$$
$$\times\hat{\mathbf{J}}^{\mathrm{T}}(\mathbf{x},\omega)\hat{\mathbf{J}}^{\mathrm{T}*}(\mathbf{x},\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega', \tag{2.7}$$

and the averaged longitudinal Poynting vector,

$$\langle \mathbf{S}^{\mathrm{L}}\rangle \sim \frac{m_{\mathrm{t}}^2\mathbf{n}}{4(2\pi)^4r^2}\frac{2\pi}{T}\int\int_{-\infty}^{+\infty} \omega^{-1}k(\omega)\delta_{(1)}(\omega-\omega';T)$$
$$\times\hat{\mathbf{J}}^{\mathrm{L}}(\mathbf{x},\omega)\hat{\mathbf{J}}^{\mathrm{L}*}(\mathbf{x},\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega' , \tag{2.8}$$

with (2.6) substituted. Here, we use a standard limit definition of the Dirac function,

$$\delta_{(1)}(\omega;T) := \frac{1}{2\pi}\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{\pi}\frac{\sin(T\omega/2)}{\omega},$$

$$\delta_{(2)}(\omega;T) := \frac{2\pi}{T}\left(\delta_{(1)}(\omega;T)\right)^2 , \tag{2.9}$$

so that $\delta_{(1)}(\omega;T\to\infty)=\delta(\omega)$, and the same for $\delta_{(2)}$. The purpose of these limit representations is to avoid ill-defined squares of δ-functions. The radiant power is obtained by integrating the flux through a sphere of radius $r \to \infty$,

$$P = P^{\mathrm{T}} + P^{\mathrm{L}}, \quad P^{\mathrm{T,L}} := r^2 \int \mathbf{n} \cdot \langle \mathbf{S}^{\mathrm{T,L}} \rangle \, \mathrm{d}\Omega \,, \tag{2.10}$$

with the solid angle element $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$.

We specialize to helical motion; the subluminal source moves with constant speed, otherwise there are no restrictions on the velocity. The ultra-relativistic limit, tachyonic synchrotron radiation from particles in circular motion, will be studied in greater detail in Sections 3–6. Tachyonic cyclotron radiation will be dealt with elsewhere; in fact, tachyon radiation by non-relativistic particles always requires quantization, cf. the beginning of Section 5. We consider a constant magnetic field $\mathbf{B} = (0, 0, B)$, $B > 0$, and a particle with constant speed υ, electric charge e and tachyonic charge q. The equations of motion read $\mathrm{d}(\gamma m \mathbf{v})/\mathrm{d}t = (e/c)\mathbf{v} \times \mathbf{B}$, where $\gamma := (1 - \upsilon^2/c^2)^{-1/2}$. Heaviside–Lorentz units are used throughout, and e is defined negative for electrons. We so find the helix

$$\mathbf{x}(t) = \left(-\frac{\upsilon_\perp}{\omega_{\mathrm{B}}} \operatorname{sign}(e) \cos(\omega_{\mathrm{B}} t), \frac{\upsilon_\perp}{\omega_{\mathrm{B}}} \sin(\omega_{\mathrm{B}} t), \ \upsilon_\| t \right) \,,$$

$$\omega_{\mathrm{B}} := |e|B/(mc\gamma), \quad \upsilon_\perp := \upsilon \sin\alpha, \quad \upsilon_\| := \upsilon\cos\alpha \,, \tag{2.11}$$

where α is the pitch angle between $\mathbf{B}$ and $\mathbf{v}$, and $0 < \alpha < \pi$. The gyrofrequency relates to the Larmor frequency as $\omega_{\mathrm{L}} = \omega_{\mathrm{B}}\gamma/2$. For circular motion in the (x, y)-plane, we have of course $\alpha = \pi/2$. We will also admit $\operatorname{sign}(e) = 0$ in (2.11), that is, planar transversal oscillations, realizable by undulators in storage rings [7], though we will not discuss this here. The notation is kept close to electromagnetic synchrotron radiation [8–11].

We introduce polar coordinates with $\mathbf{B}$ as polar axis and polar angle θ, and consider the wave vector in the (y,z)-plane, without loss of generality, so that $\mathbf{k} := k(\omega)\mathbf{n}$, $\mathbf{n} = (0, \sin\theta, \cos\theta)$, with $k(\omega)$ defined in (2.3). There is no necessity to specify $k(\omega)$ for $\omega = 0$, as there is no wave propagation in this case. The $\mathbf{e}_i$ are unit vectors along the coordinate axes, and we define two further unit vectors,

$$\boldsymbol{\varepsilon}_\| := \frac{\mathbf{e}_3 - \mathbf{n}(\mathbf{n} \cdot \mathbf{e}_3)}{|\mathbf{e}_3 - \mathbf{n}(\mathbf{n} \cdot \mathbf{e}_3)|} = (0, -\cos\theta, \sin\theta), \quad \boldsymbol{\varepsilon}_\perp := \boldsymbol{\varepsilon}_\| \times \mathbf{n} = -\mathbf{e}_1,$$

$$\boldsymbol{\varepsilon}_\| = -\boldsymbol{\varepsilon}_\perp \times \mathbf{n} \,, \tag{2.12}$$

so that $\mathbf{n}$, $\boldsymbol{\varepsilon}_\|$ and $\boldsymbol{\varepsilon}_\perp$ constitute an orthonormal triad. We may thus write the transversal and longitudinal components of the velocity as

$$\mathbf{v}^{\mathrm{T}} = \boldsymbol{\varepsilon}_\| (\upsilon_\| \sin\theta - \upsilon_\perp \cos\theta \cos(\omega_{\mathrm{B}} t)) - \boldsymbol{\varepsilon}_\perp \upsilon_\perp \operatorname{sign}(e) \sin(\omega_{\mathrm{B}} t) \,,$$

$$\mathbf{v}^{\mathrm{L}} = \mathbf{n}(\upsilon_\perp \sin\theta \cos(\omega_{\mathrm{B}} t) + \upsilon_\| \cos\theta) \tag{2.13}$$

and we also note

$$\mathbf{n} \cdot \mathbf{x}(t) = (\upsilon_\perp/\omega_{\mathrm{B}}) \sin(\omega_{\mathrm{B}} t) \sin\theta + \upsilon_\| t \cos\theta \,. \tag{2.14}$$

When studying electromagnetic synchrotron radiation, it is customary to choose a rest frame where the helical orbit appears circular. A noticeable exception is Ref. [10], and we will follow the reasoning there. The introduction of a frame comoving along the helical axis is not useful when studying superluminal radiation, as in such frames longitudinal and transversal modes can appear tangled or even advanced. We will study superluminal radiation in the comoving galaxy frame, locally realized by a Minkowskian frame in which the microwave background is isotropic, that is, Planckian with an isotropic temperature. Both the causality and the energy concept for tachyons are based on the universal cosmic time order, on the absolute space–time defined by the galaxy grid, locally manifested by the background radiations and the cosmic ether [4,12–14]. The helix (2.11) is meant in this frame, in a locally geodesic neighborhood in which the microwave radiation is isotropic.

We start by truncating the integral representation (2.6) of the current,

$$\hat{\mathbf{J}}^{\mathrm{T,L}}(\theta,\omega) = q\int_{-T/2}^{+T/2} \mathrm{d}t\mathbf{v}^{\mathrm{T,L}}\exp[\mathrm{i}(\omega t - k(\omega)\mathbf{n}\cdot\mathbf{x}(t))]\,, \tag{2.15}$$

which can be evaluated via (2.13), (2.14) and Jacobi's expansion,

$$\exp(-\mathrm{i}z\sin(\tilde{\omega}t)) = \sum_{n=-\infty}^{+\infty} J_n(z)\mathrm{e}^{-\mathrm{i}n\tilde{\omega}t}\,. \tag{2.16}$$

To this end, we use the smooth limit representation (2.9) of the δ-function and the truncated Fourier transforms

$$\int_{-T/2}^{T/2} \sin(\tilde{\omega}t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}\,\pi(\delta_{(1)}(\omega+\tilde{\omega};T) - \delta_{(1)}(\omega-\tilde{\omega};T))\,,$$
$$\int_{-T/2}^{T/2} \cos(\tilde{\omega}t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \pi(\delta_{(1)}(\omega+\tilde{\omega};T) + \delta_{(1)}(\omega-\tilde{\omega};T))\,, \tag{2.17}$$

as well as the identities $J_{n+1} - J_{n-1} = -2J_n'(z)$ and $J_{n+1} + J_{n-1} = 2(n/z)J_n(z)$. Expansion (2.16) amounts to a reordered multipole expansion, frequently used in radiation problems with periodically moving sources [11]. In this way, we arrive at

$$\hat{\mathbf{J}}^{\mathrm{T}} = q(\boldsymbol{\varepsilon}_{\|}(\upsilon_{\|}\sin\theta\tilde{J}_1 - \upsilon_{\perp}\cos\theta\tilde{J}_{\cos}) - \boldsymbol{\varepsilon}_{\perp}\upsilon_{\perp}\,\mathrm{sign}(e)\tilde{J}_{\sin})\,,$$

$$\hat{\mathbf{J}}^{\mathrm{L}} = q\mathbf{n}(\upsilon_{\perp}\sin\theta\tilde{J}_{\cos} + \upsilon_{\|}\cos\theta\tilde{J}_1)\,,$$

$$\tilde{J}_{(1,\cos,\sin)}(\theta,\omega) := \int_{-T/2}^{+T/2} \mathrm{d}t(1,\cos(\omega_{\mathrm{B}}t),\sin(\omega_{\mathrm{B}}t))\exp[\mathrm{i}(\omega t - k(\omega)\mathbf{n}\cdot\mathbf{x}(t))]$$
$$= 2\pi\sum_{n=-\infty}^{+\infty}\left(J_n(z),\frac{n}{z}J_n(z),\mathrm{i}J_n'(z)\right)\delta_{(1)}(\alpha_n;T)\,, \tag{2.18}$$

$$z(\omega) := k(\omega)(\upsilon_{\perp}/\omega_{\mathrm{B}})\sin\theta, \quad \alpha_n(\omega) := n\omega_{\mathrm{B}} - \omega + k(\omega)\upsilon_{\|}\cos\theta\,. \tag{2.19}$$

Collecting terms, we may write

$$\hat{\mathbf{J}}^{\mathrm{T,L}}(\theta,\omega) = 2\pi q\sum_{n=-\infty}^{+\infty}\boldsymbol{\lambda}_n^{\mathrm{T,L}}(z(\omega))\delta_{(1)}(\alpha_n;T)\,, \tag{2.20}$$

$$\boldsymbol{\lambda}_n^{\mathrm{T}}(z) := \boldsymbol{\varepsilon}_{\|}(\upsilon_{\|}\sin\theta - (n/z)\upsilon_{\perp}\cos\theta)J_n(z) - \mathrm{i}\boldsymbol{\varepsilon}_{\perp}\upsilon_{\perp}\,\mathrm{sign}(e)J_n'(z)\ ,$$

$$\boldsymbol{\lambda}_n^{\mathrm{L}}(z) := \mathbf{n}((n/z)\upsilon_{\perp}\sin\theta + \upsilon_{\|}\cos\theta)J_n(z)\ , \tag{2.21}$$

As for the flux vectors (2.7) and (2.8), we need to know the zeros ω_n of $\alpha_n(\omega) = 0$, cf. (2.19). For every n, there are at most two solutions,

$$\omega_n^{\pm} = \frac{n\omega_{\mathrm{B}} \pm (\upsilon_{\|}/c)\cos\theta\beta_n}{1-(\upsilon_{\|}/c)^2\cos^2\theta}, \quad \beta_n := \sqrt{n^2\omega_{\mathrm{B}}^2 + m_{\mathrm{t}}^2c^2(1-(\upsilon_{\|}/c)^2\cos^2\theta)}\ , \tag{2.22}$$

$\beta_n \geqslant 0$, and we define

$$k_n^{\pm} := \frac{\omega_n^{\pm} - n\omega_{\mathrm{B}}}{\upsilon_{\|}\cos\theta} = \frac{1}{c}\,\frac{n\omega_{\mathrm{B}}(\upsilon_{\|}/c)\cos\theta \pm \beta_n}{1-(\upsilon_{\|}/c)^2\cos^2\theta}\ . \tag{2.23}$$

Apparently, $k_n^+ > 0$ and $k_n^- < 0$, and thus only positive ω_n^+ and negative ω_n^- can be solutions, because only then $k_n^+ = k(\omega_n^+)$ and $k_n^- = k(\omega_n^-)$, respectively. Hence, for sufficiently large $|n|$, there is exactly one solution, namely ω_n^+ for positive n, and ω_n^- for negative integers. If the mass term is dominant in (2.22), there may be two solutions of $\alpha_n(\omega) = 0$ or none. Solutions $\omega_n^{\pm} = 0$ are discarded, as they do not correspond to wave modes. Hence,

$$\delta(\alpha_n) = c\sum_{\pm}(|k(\omega_n^{\pm})|/\beta_n)\theta(\pm\omega_n^{\pm})\delta(\omega - \omega_n^{\pm})\ . \tag{2.24}$$

We note $J_{-n}(-z) = J_n(z)$ and $J'_{-n}(-z) = -J'_n(z)$, define

$$z_n^{\pm} := z(\omega_n^{\pm}) = k(\omega_n^{\pm})(\upsilon_{\perp}/\omega_{\mathrm{B}})\sin\theta\ , \tag{2.25}$$

and find the symmetries

$$\omega_n^{\pm} = -\omega_{-n}^{\mp}, \quad z_n^{\pm} = -z_{-n}^{\mp}, \quad \boldsymbol{\lambda}_n^{\mathrm{T,L}*}(z) = \boldsymbol{\lambda}_{-n}^{\mathrm{T,L}}(-z), \quad \hat{\mathbf{J}}^*(\theta,\omega) = \hat{\mathbf{J}}(\theta,-\omega)\ . \tag{2.26}$$

With these preparations, we can readily compile the Poynting vectors (2.7) and (2.8),

$$\langle \mathbf{S}^{\mathrm{T}}\rangle \sim \frac{q^2\mathbf{n}}{(4\pi r)^2c}\sum_{n\in Z,\pm}\beta_n^{-1}|\omega_n^{\pm}||k(\omega_n^{\pm})|^2|\boldsymbol{\lambda}_n^{\mathrm{T}}(z_n^{\pm})|^2\theta(\pm\omega_n^{\pm})\ , \tag{2.27}$$

$$\langle \mathbf{S}^{\mathrm{L}}\rangle \sim m_{\mathrm{t}}^2c\frac{q^2\mathbf{n}}{(4\pi r)^2}\sum_{n\in Z,\pm}\beta_n^{-1}|\omega_n^{\pm}|^{-1}|k(\omega_n^{\pm})|^2|\boldsymbol{\lambda}_n^{\mathrm{L}}(z_n^{\pm})|^2\theta(\pm\omega_n^{\pm})\ , \tag{2.28}$$

$$|\boldsymbol{\lambda}_n^{\mathrm{T}}(z_n^{\pm})|^2 = (\upsilon_{\|}\sin\theta - (n/z_n^{\pm})\upsilon_{\perp}\cos\theta)^2J_n^2(z_n^{\pm}) + \mathrm{sign}^2(e)\upsilon_{\perp}^2J_n'^2(z_n^{\pm})\ , \tag{2.29}$$

$$|\boldsymbol{\lambda}_n^{\mathrm{L}}(z_n^{\pm})|^2 = ((n/z_n^{\pm})\upsilon_{\perp}\sin\theta + \upsilon_{\|}\cos\theta)^2J_n^2(z_n^{\pm})\ , \tag{2.30}$$

where $|\boldsymbol{\lambda}_n|^2 := \boldsymbol{\lambda}_n\cdot\boldsymbol{\lambda}_n^*$. In (2.27) and (2.28), we may replace $\sum_{n\in Z,\pm}$ by $\sum_{n=0,\pm} + 2\sum_{n\geqslant 1,\pm}$ without further changes, due to symmetries (2.26). In (2.29) we refrain from writing $\mathrm{sign}^2(e)=1$, since in this way the contributions of the two linear transversal polarizations (defined by $\boldsymbol{\varepsilon}_{\|,\perp}$ in (2.12) and (2.21)) to the radiated power can easily be distinguished; the terms stemming from the $\boldsymbol{\varepsilon}_\perp$-polarized component of the radiation field are proportional to $\mathrm{sign}^2(e)$. Moreover, by putting $\mathrm{sign}^2(e)=0$, we find the radiation generated by oscillating charges in undulator fields, mentioned after (2.11). The power transversally and longitudinally radiated is thus, cf. (2.10),

$$P^{\mathrm{T}}=\frac{1}{2}\frac{q^2}{4\pi c}\int_0^\pi \sum_{n\in Z,\pm}^{\mathrm{T}}\sin\theta\,\mathrm{d}\theta,\quad P^{\mathrm{L}}=\frac{1}{2}\frac{q^2}{4\pi}m_{\mathrm{t}}^2 c\int_0^\pi \sum_{n\in Z,\pm}^{\mathrm{L}}\sin\theta\,\mathrm{d}\theta\,, \tag{2.31}$$

where the summation signs stand for the series in the flux vectors (2.27) and (2.28), respectively. In the next section, we will evaluate these vectors for ultra-relativistic circular motion, and derive explicit formulas for the tachyonic spectral distributions. In Section 4, we will have a closer look at the radiant powers (2.31).

3. Tachyonic spectral densities, curvature radiation, and the longitudinal Stokes parameter

We derive the transversal and longitudinal spectral densities for tachyon radiation emitted by ultra-relativistic (subluminal) particles, $\upsilon/c\approx 1$, in circular motion. The context is given in Section 2, specialized to circular orbits, $\upsilon_\|=0$, $\upsilon_\perp=\upsilon$. The spectral modes are $\omega_n=n\omega_{\mathrm{B}}$, cf. (2.22), so that in the notation of (2.19) and (2.23),

$$k_n=\frac{\omega_{\mathrm{B}}}{\upsilon}x_n,\quad z_n:=z(\omega_n)=x_n\sin\theta,\quad \frac{x_n}{n}:=\frac{\upsilon}{c}\sqrt{1+\frac{m_{\mathrm{t}}^2c^2}{n^2\omega_{\mathrm{B}}^2}}\,. \tag{3.1}$$

Here, ω_n means $\omega^+_{n>0}$ or $\omega^-_{n<0}$, cf. after (2.23). The multipole expansion of the time averaged Poynting vectors, cf. (2.27)–(2.30), reads $\langle\mathbf{S}^{\mathrm{T,L}}\rangle=\sum_{n=1}^\infty\langle\mathbf{S}_n^{\mathrm{T,L}}\rangle$, where

$$\langle\mathbf{S}_n^{\mathrm{T}}\rangle=\frac{2\mathbf{n}q^2}{(4\pi r)^2}\frac{\omega_{\mathrm{B}}^2n^2}{c^2\upsilon}\frac{x_n}{n}|\boldsymbol{\lambda}_n^{\mathrm{T}}(z_n)|^2,\quad \langle\mathbf{S}_n^{\mathrm{L}}\rangle=\frac{2\mathbf{n}q^2}{(4\pi r)^2}\frac{m_{\mathrm{t}}^2}{\upsilon}\frac{x_n}{n}|\boldsymbol{\lambda}_n^{\mathrm{L}}(z_n)|^2\,, \tag{3.2}$$

$$|\boldsymbol{\lambda}_n^{\mathrm{T}}(z_n)|^2=\upsilon^2\frac{n^2}{x_n^2}\cot^2\theta J_n^2(x_n\sin\theta)+\upsilon^2\,\mathrm{sign}^2(e)J_n'^2(x_n\sin\theta)\,,$$

$$|\boldsymbol{\lambda}_n^{\mathrm{L}}(z_n)|^2=\upsilon^2\frac{n^2}{x_n^2}J_n^2(x_n\sin\theta)\,. \tag{3.3}$$

Carrying out the angular integration, we find the transversal and longitudinal components of the total power radiated as

$$P^{\mathrm{T,L}}=\sum_{n=1}^\infty P_n^{\mathrm{T,L}},\quad P_n^{\mathrm{T,L}}=2\pi r^2\int_0^\pi\langle\mathbf{n}\cdot\mathbf{S}_n^{\mathrm{T,L}}\rangle\sin\theta\,\mathrm{d}\theta\,. \tag{3.4}$$

More explicitly,

$$P_n^{\mathrm{T}}=\frac{q^2}{4\pi}\frac{\omega_{\mathrm{B}}^2n}{c}\frac{\upsilon}{c}\frac{n^2}{x_n^2}\left(H_\|^{\mathrm{T}}(x_n)+\mathrm{sign}^2(e)H_\perp^{\mathrm{T}}(x_n)\right)\,, \tag{3.5}$$

$$P_n^{\mathrm{L}} = \frac{q^2}{4\pi} m_{\mathrm{t}}^2 c \frac{1}{n} \frac{\upsilon}{c} \frac{n^2}{x_n^2} H^{\mathrm{L}}(x_n) , \tag{3.6}$$

where the power radiated in the transversal linear $\boldsymbol{\varepsilon}_{\|}$ and $\boldsymbol{\varepsilon}_{\perp}$ polarizations is determined by

$$H_{\|}^{\mathrm{T}}(x) := x \int_0^{\pi} \cot^2 \theta J_n^2(x \sin\theta) \sin\theta \, \mathrm{d}\theta$$

$$= 2x \int_0^{2x} z^{-1} J_{2n}(z) \, \mathrm{d}z - \int_0^{2x} J_{2n}(z) \, \mathrm{d}z , \tag{3.7}$$

$$H_{\perp}^{\mathrm{T}}(x) := \frac{x^3}{n^2} \int_0^{\pi} J_n'^2(x \sin\theta) \sin\theta \, \mathrm{d}\theta$$

$$= 2 \frac{x^2}{n^2} J_{2n}'(2x) + \frac{x^2}{n^2} \int_0^{2x} J_{2n}(z) \, \mathrm{d}z - 2x \int_0^{2x} z^{-1} J_{2n}(z) \, \mathrm{d}z , \tag{3.8}$$

respectively, and the longitudinally radiated power is defined by

$$H^{\mathrm{L}}(x) := x \int_0^{\pi} J_n^2(x \sin\theta) \sin\theta \, \mathrm{d}\theta = \int_0^{2x} J_{2n}(z) \, \mathrm{d}z , \tag{3.9}$$

cf. Appendix A for further comments on these identities. Radiation generated by transversal planar oscillations corresponds to $\mathrm{sign}^2(e) = 0$; we will always indicate $\mathrm{sign}^2(e)$ to distinguish the polarization components of the transversal radiation, cf. after (2.30).

The preceding calculations are exact, apart from the $r \to \infty$ asymptotics, which is sufficient to calculate the radiated energy. We will evaluate $P_n^{\mathrm{T,L}}$ for large n and $\upsilon/c \approx 1$. This can be done by means of the Nicolson asymptotics sketched in Appendix A. As n is large, we may pass to continuous frequencies via $\omega = n\omega_{\mathrm{B}}$, so that $P_n^{\mathrm{T,L}} \, \mathrm{d}n \approx p^{\mathrm{T,L}}(\omega) \, \mathrm{d}\omega$, with the spectral densities

$$p^{\mathrm{T,L}}(\omega) := \omega_{\mathrm{B}}^{-1} P_{n=\omega/\omega_{\mathrm{B}}}^{\mathrm{T,L}} . \tag{3.10}$$

There are two asymptotic regimes, depending on whether $x_n/n > 1$ or $x_n/n < 1$, cf. (3.1). To study that, it is almost indispensable to introduce, cf. (A.9),

$$\xi(\omega) := \kappa \frac{\omega_{\mathrm{b}}^2}{\omega^2} |1 - \omega^2/\omega_{\mathrm{b}}^2|^{3/2}, \quad \omega_{\mathrm{b}} := \gamma m_{\mathrm{t}} c, \quad \kappa := \frac{2}{3} \frac{m_{\mathrm{t}} c}{\gamma^2 \omega_{\mathrm{B}}} . \tag{3.11}$$

First we consider the case $x_n/n \geqslant 1$, which means frequencies satisfying $\omega \leqslant \omega_{\mathrm{b}}$. In this regime, the spectral densities follow from (3.5) and (3.6), with substitutions (A.7) and (A.8). The transversal density is assembled as

$$p^{\mathrm{T}}(\omega) = p_{\|}^{\mathrm{T}}(\omega) + \mathrm{sign}^2(e) p_{\perp}^{\mathrm{T}}(\omega) , \tag{3.12}$$

where, for $\omega \leqslant \omega_{\mathrm{b}}$,

$$p_{\|,\perp}^{\mathrm{T}}(\omega \leqslant \omega_{\mathrm{b}}) = \frac{1}{2} \frac{q^2}{4\pi} \frac{m_{\mathrm{t}}^2 c\omega}{\omega^2 + m_{\mathrm{t}}^2 c^2} \left(1 - \frac{\omega^2}{\omega_{\mathrm{b}}^2}\right) (1 + F_0(\xi) \mp G_0(\xi)) , \tag{3.13}$$

$$F_0(\xi) := -\frac{1}{3}\int_\xi^\infty (J_{-5/3}(x) + J_{5/3}(x))\,\mathrm{d}x, \quad G_0(\xi) := \frac{1}{3}(J_{-2/3}(\xi) - J_{2/3}(\xi)) ,$$

$$F_0(\xi) = 2G_0(\xi) - L_0(\xi), \quad L_0(\xi) := \frac{1}{3}\int_\xi^\infty (J_{-1/3}(x) + J_{1/3}(x))\,\mathrm{d}x , \tag{3.14}$$

with ξ and ω_b in (3.11). The lower plus-sign in (3.13) refers to the $\perp$-polarization. The longitudinal spectral density reads

$$p^\mathrm{L}(\omega \leqslant \omega_\mathrm{b}) = \frac{q^2}{4\pi}\frac{m_\mathrm{t}^2 c\omega}{\omega^2 + m_\mathrm{t}^2 c^2}(1 - L_0(\xi)) . \tag{3.15}$$

The second case, $x_n/n \leqslant 1$, refers to the upper frequency range, $\omega \geqslant \omega_\mathrm{b}$. Explicit expressions for the spectral densities again follow from (3.5) and (3.6), but now with substitutions (A.17) and (A.18). The transversal density is composed as in (3.12), with

$$p^\mathrm{T}_{\parallel,\perp}(\omega \geqslant \omega_\mathrm{b}) := \frac{1}{2}\frac{q^2}{4\pi}\frac{m_\mathrm{t}^2 c\omega}{\omega^2 + m_\mathrm{t}^2 c^2}\left(\frac{\omega^2}{\omega_\mathrm{b}^2} - 1\right)(F_\infty(\xi) \mp G_\infty(\xi)) , \tag{3.16}$$

$$F_\infty(\xi) := \frac{1}{\sqrt{3}\pi}\int_\xi^\infty K_{5/3}(x)\,\mathrm{d}x, \quad G_\infty(\xi) := \frac{1}{\sqrt{3}\pi}K_{2/3}(\xi) ,$$

$$F_\infty(\xi) = 2G_\infty(\xi) - L_\infty(\xi), \quad L_\infty(\xi) := \frac{1}{\sqrt{3}\pi}\int_\xi^\infty K_{1/3}(x)\,\mathrm{d}x , \tag{3.17}$$

where ξ and ω_b are defined in (3.11). Finally, the longitudinal density in this frequency range,

$$p^\mathrm{L}(\omega \geqslant \omega_\mathrm{b}) = \frac{q^2}{4\pi}\frac{m_\mathrm{t}^2 c\omega}{\omega^2 + m_\mathrm{t}^2 c^2}L_\infty(\xi) . \tag{3.18}$$

The asymptotic expansions of the spectral functions $F_{0,\infty}$, $G_{0,\infty}$ and $L_{0,\infty}$ are listed in (A.10), (A.11), (A.19), and (A.20).

The massless limit is easily recovered, electromagnetic radiation that is, where ξ reduces to $\xi_{m_\mathrm{t}=0} = \omega/\omega_\mathrm{c}$, with the critical frequency $\omega_\mathrm{c} := (3/2)\omega_\mathrm{B}\gamma^3$. As $\omega_\mathrm{b} = \gamma m_\mathrm{t} c \to 0$, we find from (3.16),

$$p^\mathrm{ph}_{\parallel,\perp}(\omega) = \frac{1}{2}\frac{q^2}{4\pi}\frac{\omega}{\gamma^2 c}(F_\infty(\omega/\omega_\mathrm{c}) \mp G_\infty(\omega/\omega_\mathrm{c})) , \tag{3.19}$$

with q replaced by the electric charge, of course.

Even though densities (3.13) and (3.16) have a very different shape in the upper and lower spectral range, they smoothly join at ω_b, in fact analytically, and the same holds for the longitudinal densities (3.15) and (3.18). To see this, we define

$$\eta(\omega) := (3\kappa/2)^{2/3}(\omega_\mathrm{b}/\omega)^{4/3}\left(\frac{\omega^2}{\omega_\mathrm{b}^2} - 1\right) , \tag{3.20}$$

so that $\xi = (2/3)|\eta|^{3/2}$, cf. (3.11), and we find, via (A.22),

$$p^{\mathrm{T}}_{\|,\perp}(\omega) = \frac{1}{2}\frac{q^2}{4\pi}\frac{m_{\mathrm{t}}^2 c\omega}{\omega^2 + m_{\mathrm{t}}^2 c^2}\left(1 - \frac{\omega^2}{\omega_{\mathrm{b}}^2}\right)\left(\int_\eta^\infty \mathrm{Ai}(x)\,\mathrm{d}x + (2 \mp 1)\frac{1}{\eta}\,\mathrm{Ai}'(\eta)\right) ,$$

$$p^{\mathrm{L}}(\omega) = \frac{q^2}{4\pi}\frac{m_{\mathrm{t}}^2 c\omega}{\omega^2 + m_{\mathrm{t}}^2 c^2}\int_\eta^\infty \mathrm{Ai}(x)\,\mathrm{d}x , \qquad (3.21)$$

valid throughout the spectral range. The lower plus-sign again refers to the $\perp$-polarization. In the next section, we will integrate these densities to obtain the radiant power. This will be done asymptotically, with κ as expansion parameter, cf. (3.11).

We return to the lower spectral range, $\omega \leqslant \omega_{\mathrm{b}}$, cf. (3.13) and (3.15). In the limit $\kappa \to \infty$, i.e., for $\xi \to \infty$ (at a fixed frequency), the spectral functions F_0, G_0 and L_0 all vanish, which happens in the limit of infinite gyroradius, since $\kappa \propto R$, as will be shown in (4.2). This suggests to split the spectral densities into

$$p^{\mathrm{T}}_{\|,\perp}(\omega \leqslant \omega_{\mathrm{b}}) = p^{\mathrm{T,lin}}/2 - p^{\mathrm{T,curv}}_{\|,\perp}, \quad p^{\mathrm{L}}(\omega \leqslant \omega_{\mathrm{b}}) = p^{\mathrm{L,lin}} - p^{\mathrm{L,curv}} , \qquad (3.22)$$

where the densities $p^{\mathrm{T,lin}}$ and $p^{\mathrm{L,lin}}$ stand for the tachyon radiation generated by a charge in linear uniform motion (in the ultra-relativistic limit, that is) [6],

$$p^{\mathrm{T,lin}}(\omega) := \frac{q^2}{4\pi}\frac{m_{\mathrm{t}}^2 c\omega}{\omega^2 + m_{\mathrm{t}}^2 c^2}\left(1 - \frac{\omega^2}{\omega_{\mathrm{b}}^2}\right), \quad p^{\mathrm{L,lin}}(\omega) := \frac{q^2}{4\pi}\frac{m_{\mathrm{t}}^2 c\omega}{\omega^2 + m_{\mathrm{t}}^2 c^2} . \qquad (3.23)$$

The curvature radiation subtracted in (3.22) reads

$$p^{\mathrm{T,curv}}_{\|,\perp}(\omega) : = -(p^{\mathrm{T,lin}}/2)(F_0 \mp G_0), \quad p^{\mathrm{L,curv}}(\omega) := p^{\mathrm{L,lin}} L_0 , \qquad (3.24)$$

where the upper minus-sign in (3.24) refers to $p^{\mathrm{T,curv}}_{\|}$. The Lorentz factor of the charge enters in the transversal linear density $p^{\mathrm{T,lin}}$ via ω_{b}. It also enters in $p^{\mathrm{L,lin}}$, again by ω_{b}, which is the cutoff frequency for uniform motion. A uniformly moving ultra-relativistic charge can only radiate frequencies $\omega \leqslant \omega_{\mathrm{b}}$. The radiation in the upper frequency range, $\omega \geqslant \omega_{\mathrm{b}}$, is pure curvature radiation, the spectral functions F_∞, G_∞ and L_∞ vanish in the limit of zero orbital curvature, $\kappa \to \infty$. The densities $p^{\mathrm{T,L}}_{\|,\perp}$ in (3.22) are positive definite, but not so the curvature terms $p^{\mathrm{T,curv}}$ and $p^{\mathrm{L,curv}}$ in (3.24), which oscillate for large ξ, cf. (A.11). Densities (3.21) are positive throughout the spectral range, of course. In the next section, we will show that the curvature terms in densities (3.22) give negative contributions to the radiated power, nearly equal in magnitude to the power radiated as curvature radiation in the upper frequency range.

Once the spectral densities are known, the transversal and longitudinal polarization functions, $\Pi^{\mathrm{T,L}}_\omega := p^{\mathrm{T,L}}/(p^{\mathrm{T}} + p^{\mathrm{L}})$, are readily assembled. We find in the lower spectral range,

$$\Pi^{\mathrm{T}}_{\omega \leqslant \omega_{\mathrm{b}}} = \frac{X_0^2}{1 + X_0^2}, \quad \Pi^{\mathrm{L}}_{\omega \leqslant \omega_{\mathrm{b}}} = \frac{1}{1 + X_0^2}, \quad X_0^2(\omega) := \left(1 - \frac{\omega^2}{\omega_{\mathrm{b}}^2}\right)\frac{1 + F_0}{1 - L_0} , \qquad (3.25)$$

and in the high-frequency regime,

$$\Pi^{\rm T}_{\omega\geqslant\omega_{\rm b}} = \frac{X_\infty^2}{1+X_\infty^2}, \quad \Pi^{\rm L}_{\omega\geqslant\omega_{\rm b}} = \frac{1}{1+X_\infty^2}, \quad X_\infty^2(\omega) := \left(\frac{\omega^2}{\omega_{\rm b}^2}-1\right)\frac{F_\infty}{L_\infty}\ . \tag{3.26}$$

The argument in the spectral functions is always $\xi(\omega)$. We will content ourselves with the integrated versions of these ratios, and replace the densities in $\Pi_\omega^{\rm T,L}$ by the powers radiated in the respective polarizations, cf. (4.17).

Tachyon radiation can be longitudinally polarized, and the longitudinal component may even overpower the transversal radiation. I therefore conclude this section with the Stokes parameter for longitudinal radiation. Tachyonic E and B-fields are related to the vector potential in the usual way, by $\mathbf{E} = (1/c)(\nabla A_0 - \partial\mathbf{A}/t)$ and $\mathbf{B} = {\rm rot}\,\mathbf{A}$, cf. Ref. [15]. We consider a Fourier mode

$$\mathbf{E}_\mathbf{k}(\mathbf{x},t) = \sum_{n=1}^{3} \boldsymbol{\varepsilon}_{\mathbf{k},n} a_{\mathbf{k},n}(t) \exp({\rm i}(\mathbf{kx}-\omega t))\ , \tag{3.27}$$

the $\boldsymbol{\varepsilon}_{\mathbf{k},1}$ and $\boldsymbol{\varepsilon}_{\mathbf{k},2}$ are real unit vectors (linear polarization vectors) orthogonal to $\boldsymbol{\varepsilon}_{\mathbf{k},3} := \mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$, so that the $\boldsymbol{\varepsilon}_{\mathbf{k},n}$ constitute an orthonormal triad. The $a_{\mathbf{k},n} := r_{\mathbf{k},n}(t)$ $\exp({\rm i}\varphi_{\mathbf{k},n}(t))$ are complex amplitudes, slowly varying in time as compared to the phase factor in (3.27). Defining $E_{\mathbf{k},m} := \boldsymbol{\varepsilon}_{\mathbf{k},m}\cdot\mathbf{E}_\mathbf{k}$, the polarization can be determined from the time averages $\langle E_{\mathbf{k},m}E^*_{\mathbf{k},n}\rangle \equiv \langle a_{\mathbf{k},m}a^*_{\mathbf{k},n}\rangle$ taken over the period $2\pi/\omega$. In the following we will suppress the subscript $\mathbf{k}$. The transversal degrees, $m,n=1,2$, are settled by the Stokes parametrization

$$\langle E_m E_n^*\rangle = \frac{1}{2}\left(s_{\rm T}\delta_{mn} + \begin{pmatrix} s_1 & s_2-{\rm i}s_3 \\ s_2+{\rm i}s_3 & -s_1 \end{pmatrix}\right)\ ,$$
$$s_{\rm T} := \langle|a_1|^2\rangle + \langle|a_2|^2\rangle, \quad s_1 := 2\langle|a_1|^2\rangle - s_{\rm T} = \langle r_1^2\rangle - \langle r_2^2\rangle\ ,$$
$$s_2+{\rm i}s_3 := 2\langle a_2 a_1^*\rangle = 2\langle r_1 r_2 \exp({\rm i}(\varphi_2-\varphi_1))\rangle\ . \tag{3.28}$$

Hence, ${\rm Tr}\langle E_m E_n^*\rangle = s_{\rm T}$, and

$$\det\langle E_m E_n^*\rangle = (s_T^2/4)(1-\Pi^2_{\rm trans}), \quad \Pi^2_{\rm trans} := (s_1^2+s_2^2+s_3^2)/s_{\rm T}^2\ , \tag{3.29}$$

$0 \leqslant \Pi_{\rm trans} \leqslant 1$. Complete polarization, $\Pi_{\rm trans}=1$, in the transversal plane is achieved if the $a_{\mathbf{k},n}$ are time independent, and totally unpolarized radiation, $\Pi_{\rm trans}=0$, is indicated by vanishing $s_{1,2,3}$, so that no direction is preferred in the polarization tensor (3.28). The parameters $s_{{\rm T},1,2,3}$ account for the transversal component only, a fifth parameter is needed for the longitudinal radiation,

$$s_{\rm L} := \frac{\omega^2}{m_{\rm t}^2c^2}\langle|a_3|^2\rangle\ . \tag{3.30}$$

The real tachyonic field strength defined by (3.27) is $2\,{\rm Re}\,\mathbf{E}_\mathbf{k}$, cf. (2.1); the time averaged (over the period) transversal and longitudinal energy densities of this nearly plane wave are $\langle\rho_{\rm E}^{\rm T}\rangle \sim 2s_{\rm T}$ and $\langle\rho_{\rm E}^{\rm L}\rangle \sim 2s_{\rm L}$, respectively [4,13]. The transversal and longitudinal degrees of polarization read accordingly $\Pi^{\rm T,L} = s_{\rm T,L}/(s_{\rm T}+s_{\rm L})$, which explains the definition of $s_{\rm L}$. For instance, we may assume complete polarization in the transversal plane, $\Pi_{\mathrm{trans}} = 1$. If two of the amplitudes r_n vanish, we have linear polarization. If one of them vanishes and the other two are equal in magnitude with a phase difference of $\pm\pi/2$, the wave is circularly polarized. The longitudinal component does not show in the four transversal parameters; a field strength rotating orthogonally to the transversal plane can still be interpreted as linear transversal on the basis of $s_{\mathrm{T},1,2,3}$ only, erroneously, as one ignores the longitudinal energy density. If the superluminal velocity is close to the speed of light, the tachyon mass can be neglected in the dispersion relation, cf. (4.22), but high-energy tachyons can still be discerned from photons by their longitudinal polarization. In the examples discussed in Sections 5 and 6, the longitudinal radiation is always more pronounced than the transversal counterpart. Differential cross sections are perhaps the most practical means to discriminate longitudinal radiation from transversal tachyons and photons. Ionization cross sections have been scrutinized to that effect in Ref. [13]. The polarization of the ionizing radiation affects the angular maxima, the peaks in the transversal and longitudinal cross sections occur at different scattering angles.

4. Radiant power, tachyonic number counts, and spectral indices

We will study the tachyonic power emitted in the three polarizations, based on the ultra-relativistic spectral densities derived in the previous section, that is, the transversal densities $p^{\mathrm{T}}_{\|,\perp}(\omega)$ in (3.13) and (3.16), and the longitudinal $p^{\mathrm{L}}(\omega)$ in (3.15) and (3.18). We will discuss the integral number counts (tachyons per unit time) in the respective polarizations, and compare with electromagnetic synchrotron radiation. Finally, we will discuss tachyonic spectral slopes generated by averages over electronic source populations.

We start with some estimates concerning the helix (2.11); the notation summarized or introduced here will also be used in the tables of Section 5. Gyroradius and gyrofrequency relate as $R = \upsilon/\omega_{\mathrm{B}}$, where $\omega_{\mathrm{B}} = eB/(\gamma mc)$, $e > 0$, so that

$$\hbar\omega_{\mathrm{B}}[\mathrm{keV}] \approx 1.973\,\frac{\upsilon}{c}\,\frac{1}{R[\text{Å}]}, \quad \hbar\omega_{\mathrm{B}}[\mathrm{eV}] \approx 5.916\,\frac{B[\mathrm{kG}]}{E[\mathrm{eV}]}\,. \tag{4.1}$$

We write the Lorentz factor as $\gamma = E/mc^2$, where E and m denote energy and mass of the gyrating subluminal particle, usually electrons or positrons, so that E and m stand for electron energy and mass, and $eB = (\upsilon/c)E/R$. In the ultra-relativistic limit, we approximate $\upsilon \approx c$ whenever possible, e.g., $\omega_{\mathrm{B}} \approx c/R$. The following discussion also applies to protonic source populations, after some obvious rescaling with the electron–proton mass ratio. We use the Heaviside–Lorentz system; $e^2/(4\pi\hbar c) =: \alpha_{\mathrm{e}} \approx 1/137$ and $q^2/(4\pi\hbar c) =: \alpha_{\mathrm{q}} \approx 1.0 \times 10^{-13}$ are the electric and tachyonic fine structure constants. The tachyon mass $m_{\mathrm{t}} \approx m/238 \approx 2.15\ \mathrm{keV}/c^2$ gives a reduced Compton wave length of $\hbar/(m_{\mathrm{t}}c) \approx 0.92$ Å; the quotient of tachyonic and electric fine structure constant reads $\alpha_{\mathrm{q}}/\alpha_{\mathrm{e}} \approx 1.4 \times 10^{-11}$, all inferred from Lamb shifts in hydrogenic systems [5]. We restore the natural units, $m_{\mathrm{t}} \to m_{\mathrm{t}}c/\hbar \approx 1.09 \times 10^{8}\ \mathrm{cm}^{-1}$, cf. the beginning of Section 2, so that the break frequency reads $\omega_{\mathrm{b}} = \gamma m_{\mathrm{t}} c^2/\hbar$, cf. (3.11). In the subsequent asymptotics, we will use κ in (3.11) as expansion parameter, which is the quotient of break frequency and critical photon frequency, $\omega_c = 3\omega_B\gamma^3/2$, cf. (3.19), relating to the bending radius as

$$\kappa = \frac{\omega_b}{\omega_c} \approx \frac{2}{3}\frac{m_t c}{\hbar}\frac{R}{\gamma^2} . \tag{4.2}$$

In the examples discussed in the following sections, κ will always be large, and the asymptotic expansions will be in $\kappa^{-2/3}$, cf. (A.25). The opposite limit, $\kappa \to 0$, can be realized in the surface magnetic fields of pulsars, but we won't consider this here. The peak of the electromagnetic spectral distribution (3.19) is located approximately at $0.286\omega_c$, cf. Ref. [16], but the analytically defined ω_c is the customary reference value for the location of the bulk of the photon distribution. The tachyonic curvature radiation is peaked at the break frequency ω_b, where the linear densities (3.23) terminate, cf. the discussions following (3.24) and (4.14).

We turn to the transversal power radiated in the low-frequency regime, $\omega \leqslant \omega_b = \gamma m_t c$, cf. (3.12)–(3.15). This power can be split into polarization components like the spectral densities,

$$P^{\mathrm{T}}_{\leqslant\omega_b} := P^{\mathrm{T}}_{\parallel} + P^{\mathrm{T}}_{\perp}, \quad P^{\mathrm{T}}_{\parallel,\perp} := \int_0^{\omega_b} p^{\mathrm{T}}_{\parallel,\perp}(\omega)\,\mathrm{d}\omega . \tag{4.3}$$

Alternatively, we may decompose $P^{\mathrm{T}}_{\leqslant\omega_b}$ into a linear and a curvature component according to (3.22)–(3.24),

$$P^{\mathrm{T}}_{\leqslant\omega_b} = P^{\mathrm{T,lin}} - P^{\mathrm{T,curv}}, \quad P^{\mathrm{T,curv}} := P^{\mathrm{T,curv}}_{\parallel} + P^{\mathrm{T,curv}}_{\perp} ,$$

$$P^{\mathrm{T,lin}} := \int_0^{\omega_b} p^{\mathrm{T,lin}}(\omega)\,\mathrm{d}\omega, \quad P^{\mathrm{T,curv}}_{\parallel,\perp} := \int_0^{\omega_b} p^{\mathrm{T,curv}}_{\parallel,\perp}(\omega)\,\mathrm{d}\omega , \tag{4.4}$$

so that $P^{\mathrm{T}}_{\parallel,\perp} = P^{\mathrm{T,lin}}/2 - P^{\mathrm{T,curv}}_{\parallel,\perp}$. The respective number counts such as $N^{\mathrm{T}}_{\leqslant\omega_b}$ are defined in the same way, with the spectral densities divided by $\hbar\omega$. The power stemming from the linear transversal density reads

$$P^{\mathrm{T,lin}} \sim \alpha_q \frac{m_t^2 c^4}{\hbar}\left(\log\gamma - \frac{1}{2}\right), \quad \alpha_q \frac{m_t^2 c^4}{\hbar} \approx 0.70\ \mathrm{GeV\ s^{-1}} , \tag{4.5}$$

which is the leading order in the ultra-relativistic $1/\gamma$-expansion of the first integral in (4.4); the second integral gives the transversal curvature radiation in this frequency range,

$$P^{\mathrm{T,curv}}_{\parallel,\perp} \sim \frac{1}{2}\frac{\alpha_q m_t^2 c^4}{\hbar\kappa^{4/3}}\frac{(3 \mp 2)}{9\cdot 2^{2/3}\Gamma(2/3)}, \quad P^{\mathrm{T,curv}}[\mathrm{GeV\ s^{-1}}] \approx \frac{0.11}{\kappa^{4/3}} , \tag{4.6}$$

up to terms of $O(\kappa^{-2/3})$, cf. (A.23)–(A.27). The lower plus-sign in (4.6) refers to the $\perp$-polarization, and $P^{\mathrm{T,curv}}$ is the total transversally polarized curvature radiation as defined in (4.4). The linear power $P^{\mathrm{T,lin}}$ in (4.5) is the residual radiation in the limit of infinite gyroradius; this is the radiation emitted by an ultra-relativistic charge in uniform motion [6]. The curvature correction, $P^{\mathrm{T,curv}}_{\parallel,\perp}$, is generated by the $F_0(\xi)$ and $G_0(\xi)$-terms in (3.13), and tends to reduce the radiation in the low-frequency regime, cf. (4.4) and the discussion following (3.24).

The longitudinal radiant power can likewise be decomposed into a linear and a curvature term,

$$P^{\mathrm{L}}_{\leqslant\omega_{\mathrm{b}}} := \int_0^{\omega_{\mathrm{b}}} p^{\mathrm{L}}(\omega)\,\mathrm{d}\omega = P^{\mathrm{L,lin}} - P^{\mathrm{L,curv}} ,$$

$$P^{\mathrm{L,lin}} := \int_0^{\omega_{\mathrm{b}}} p^{\mathrm{L,lin}}(\omega)\,\mathrm{d}\omega, \quad P^{\mathrm{L,curv}} := \int_0^{\omega_{\mathrm{b}}} p^{\mathrm{L,curv}}(\omega)\,\mathrm{d}\omega . \tag{4.7}$$

An elementary integration gives the ultra-relativistic linear power,

$$P^{\mathrm{L,lin}} \sim \alpha_{\mathrm{q}} \frac{m_{\mathrm{t}}^2 c^4}{\hbar} \log\gamma , \tag{4.8}$$

up to $\mathrm{O}(\gamma^{-1})$ like in (4.5). The $\kappa^{-2/3}$-expansion of the longitudinal curvature radiation can be assembled from (A.23)–(A.27). In leading order,

$$P^{\mathrm{L,curv}} \sim \frac{\alpha_{\mathrm{q}} m_{\mathrm{t}}^2 c^4 \hbar^{-1} \kappa^{-2/3}}{3\cdot 2^{1/3}\Gamma(1/3)}, \quad P^{\mathrm{L,curv}}[\mathrm{GeV\ s^{-1}}] \approx \frac{0.069}{\kappa^{2/3}} . \tag{4.9}$$

Here and in (4.6), we use $2^{2/3}\Gamma(2/3)\approx 2.15$ and $2^{1/3}\Gamma(1/3)\approx 3.375$.

The transversal low-frequency number counts are defined like the power components in (4.3) and (4.4), with the integrands divided by $\hbar\omega$. The transversal count is assembled as $N^{\mathrm{T}}_{\leqslant\omega_{\mathrm{b}}} = N^{\mathrm{T,lin}} - N^{\mathrm{T,curv}}$, where $N^{\mathrm{T,curv}} = N^{\mathrm{T,curv}}_{\parallel} + N^{\mathrm{T,curv}}_{\perp}$, and we find

$$N^{\mathrm{T,lin}} \sim \alpha_{\mathrm{q}} \frac{m_{\mathrm{t}} c^2}{\hbar}\frac{\pi}{2}, \quad N^{\mathrm{T,lin}} \approx 5.1\times 10^5\ \mathrm{s}^{-1} , \tag{4.10}$$

$$N^{\mathrm{T,curv}}_{\parallel,\perp} \sim \frac{P^{\mathrm{T,curv}}_{\parallel,\perp}}{\hbar\omega_{\mathrm{b}}}, \quad N^{\mathrm{T,curv}}[\mathrm{s}^{-1}] \approx \frac{5.1\times 10^4}{\gamma\kappa^{4/3}} . \tag{4.11}$$

The longitudinal low-frequency count reads $N^{\mathrm{L}}_{\leqslant\omega_{\mathrm{b}}} = N^{\mathrm{L,lin}} - N^{\mathrm{L,curv}}$, with $N^{\mathrm{L,lin}} \sim N^{\mathrm{T,lin}}$, cf. (4.10), and

$$N^{\mathrm{L,curv}} \sim \frac{P^{\mathrm{L,curv}}}{\hbar\omega_{\mathrm{b}}}, \quad N^{\mathrm{L,curv}}[\mathrm{s}^{-1}] \approx \frac{3.2\times 10^4}{\gamma\kappa^{2/3}} . \tag{4.12}$$

The $\kappa^{-2/3}$-expansion of the powers and count rates beyond the leading order is sketched in (A.23)–(A.27).

The high-frequency regime, $\omega \geqslant \omega_{\mathrm{b}}$, is determined by the spectral densities (3.16)–(3.18) defining the radiant powers

$$P^{\mathrm{T}}_{\geqslant\omega_{\mathrm{b}}} := P^{\mathrm{T}}_{\parallel} + P^{\mathrm{T}}_{\perp}, \quad P^{\mathrm{T}}_{\parallel,\perp} := \int_{\omega_{\mathrm{b}}}^{\infty} p^{\mathrm{T}}_{\parallel,\perp}(\omega)\,\mathrm{d}\omega,$$

$$P^{\mathrm{L}}_{\geqslant\omega_{\mathrm{b}}} := \int_{\omega_{\mathrm{b}}}^{\infty} p^{\mathrm{L}}(\omega)\,\mathrm{d}\omega . \tag{4.13}$$

This is all curvature radiation, there is no linear contribution in contrast to the low-frequency regime. The integral number counts $N^{\mathrm{T,L}}_{\geqslant\omega_{\mathrm{b}}}$ are defined as in (4.13), with the densities divided by $\hbar\omega$. The $\kappa^{-2/3}$-expansion of these integrals is outlined in

(A.28)–(A.31), and we find in leading order,

$$P^{\mathrm{T}}_{\geqslant\omega_{\mathrm{b}}} \sim P^{\mathrm{T,curv}}, \quad P^{\mathrm{L}}_{\geqslant\omega_{\mathrm{b}}} \sim P^{\mathrm{L,curv}}, \quad N^{\mathrm{T}}_{\geqslant\omega_{\mathrm{b}}} \sim N^{\mathrm{T,curv}}, \quad N^{\mathrm{L}}_{\geqslant\omega_{\mathrm{b}}} \sim N^{\mathrm{L,curv}}\,, \tag{4.14}$$

where $P^{\mathrm{T,L,curv}}$ is the low-frequency curvature radiation calculated in (4.6) and (4.9), and $N^{\mathrm{T,L,curv}}$ is the corresponding low-frequency count in (4.11) and (4.12). This equivalence of high- and low-frequency curvature radiation also persists for the individual transversal polarizations, $P^{\mathrm{T}}_{\parallel,\perp} \sim P^{\mathrm{T,curv}}_{\parallel,\perp}$, with $P^{\mathrm{T}}_{\parallel,\perp}$ in (4.13) and $P^{\mathrm{T,curv}}_{\parallel,\perp}$ in (4.6). Relations (4.14) also hold in next to leading order in the $\kappa^{-2/3}$-expansion, but beyond that the asymptotic series start to differ, as can be seen by comparing the expansions in (A.23)–(A.27) and (A.28)–(A.31). Nevertheless, in leading order this symmetry holds, and it means that the power radiated in the upper frequency range is more or less drained from the linear component of the radiation at low frequencies, cf. (4.4) and (4.7). The curvature radiation vanishes for $\kappa \to \infty$ in both regimes; for large κ, the bulk of the curvature radiation is restricted to a small frequency range centered at the break frequency ω_{b}, roughly defined by $\xi(\omega) \leqslant 1$, cf. (3.11), (A.11) and (A.20). In the low-frequency regime, the curvature term in the spectral densities is not exponentially damped but oscillating, so that it averages itself out when integrated over a frequency range where $\xi(\omega) \gg 1$; the asymptotic expansion in (A.28)–(A.31) is based on that. For large κ, the peak frequency of the spectral distributions (3.22) is $\omega_{\mathrm{peak}} \sim m_{\mathrm{t}}c^2/\hbar \approx 3.27 \times 10^{18}$ Hz, determined by the linear densities only, and so are the tachyonic mean frequencies, $\langle \omega^{\mathrm{T,L}} \rangle := P^{\mathrm{T,L}}/N^{\mathrm{T,L}}$,

$$\langle \omega^{\mathrm{T}} \rangle \sim \frac{2}{\pi}\,\omega_{\mathrm{peak}}\left(\log\gamma - \frac{1}{2}\right), \quad \langle \omega^{\mathrm{L}} \rangle \approx \frac{2}{\pi}\,\omega_{\mathrm{peak}}\log\gamma\,. \tag{4.15}$$

In the tables of Section 5, we will compare tachyonic with electromagnetic synchrotron radiation. To this end, we quote the photonic synchrotron power, $P^{\mathrm{ph}} = P^{\mathrm{ph}}_{\parallel} + P^{\mathrm{ph}}_{\perp}$, the photon count in the usual linear polarizations, and the photonic mean energy [11],

$$P^{\mathrm{ph}}_{\parallel,\perp} := \int_0^\infty p^{\mathrm{ph}}_{\parallel,\perp}(\omega)\,\mathrm{d}\omega \sim \frac{(4 \mp 3)}{12}\,\alpha_{\mathrm{e}}\hbar\omega_{\mathrm{B}}^2\gamma^4\,,$$

$$N^{\mathrm{ph}}_{\parallel,\perp} \sim \frac{(5 \mp 3)}{4\sqrt{3}}\,\alpha_{\mathrm{e}}\omega_{\mathrm{B}}\gamma, \quad \langle \omega^{\mathrm{ph}} \rangle \approx \frac{4}{5\sqrt{3}}\,\omega_{\mathrm{B}}\gamma^3 \approx \frac{8\omega_{\mathrm{c}}}{15\sqrt{3}}\,, \tag{4.16}$$

with $\omega_{\mathrm{B}} \approx c/R$, cf. (4.1). The photon density is defined in (3.19) and integrated in (A.30) and (A.31).

Degrees of polarization, $\Pi^{\mathrm{T,L}} := P^{\mathrm{T,L}}/(P^{\mathrm{T}} + P^{\mathrm{L}})$, are defined analogously to the polarization functions in (3.25) and (3.26), with the spectral densities replaced by the radiant powers $P^{\mathrm{T,L}}_{\leqslant\omega_{\mathrm{b}}}$ or $P^{\mathrm{T,L}}_{\geqslant\omega_{\mathrm{b}}}$,

$$\Pi^{\mathrm{T}}_{\leqslant\omega_{\mathrm{b}}} \sim \frac{2\log\gamma - 1}{4\log\gamma - 1}, \quad \Pi^{\mathrm{L}}_{\leqslant\omega_{\mathrm{b}}} \sim \frac{2\log\gamma}{4\log\gamma - 1}\,,$$

$$\Pi^{\mathrm{T}}_{\geqslant\omega_{\mathrm{b}}} = 1 - \Pi^{\mathrm{L}}_{\geqslant\omega_{\mathrm{b}}} \sim \frac{2^{1/3}\Gamma(1/3)}{2^{2/3}\Gamma(2/3)}\,\frac{1}{\kappa^{2/3}} \approx \frac{1.57}{\kappa^{2/3}}\,. \tag{4.17}$$

In the low-frequency regime, the linear radiation is much more intense than the curvature radiation, cf. (4.5)–(4.9), and the polarization $\Pi^{\mathrm{T,L}}_{\leqslant\omega_\mathrm{b}}$ is thus determined by the linear powers (4.5) and (4.8), for large κ, that is. In the high-frequency range, the longitudinal curvature radiation overpowers the transversal one by a factor $\propto \kappa^{2/3}$, cf. (4.6) and (4.9), which shows in $\Pi^{\mathrm{T}}_{\geqslant\omega_\mathrm{b}}$. In the transversal plane, we find in both regimes $P_{\perp}^{\mathrm{T,curv}} : P_{\parallel}^{\mathrm{T,curv}} \sim 5:1$, cf. (4.6) and (4.14), as compared with the photonic ratio $P_{\perp}^{\mathrm{ph}} : P_{\parallel}^{\mathrm{ph}} = 7:1$.

The tables in Section 5 are compiled with the foregoing formulas; we introduce dimensionless quantities, $E_0 := E[\mathrm{GeV}]$ (electron energy), $B_0 := B[\mathrm{kG}]$, and $R_0 := R[\mathrm{m}]$, so that

$$R_0 \approx 33.36 E_0 B_0^{-1}, \quad \gamma \approx 1957 E_0, \quad \omega_\mathrm{B}[\mathrm{GHz}] \approx 0.2998 R_0^{-1}\ , \tag{4.18}$$

where we used the relations following (4.1) as well as $1\ \mathrm{kG}\cdot e \approx 2.998\times10^{-4}\ \mathrm{GeV\ cm^{-1}}$. The expansion parameter (4.2) scales as

$$\kappa \approx 1.89\times10^{3} E_0^{-2} R_0 \approx 6.32\times10^{4} E_0^{-1} B_0^{-1}\ , \tag{4.19}$$

the tachyonic powers and number counts in (4.5)–(4.12) depend on κ and γ only. The critical frequency, the break and peak frequencies, and the corresponding energies scale as

$$\begin{aligned}
&\nu_\mathrm{c}[10^{18}\ \mathrm{Hz}] \approx 0.536 E_0^3 R_0^{-1} \approx 1.61\times10^{-2} E_0^2 B_0\ ,\\
&\hbar\omega_\mathrm{c}[\mathrm{keV}] \approx 2.22 E_0^3 R_0^{-1} \approx 6.65\times10^{-2} E_0^2 B_0\ .\\
&\nu_\mathrm{b}[10^{18}\ \mathrm{Hz}] \approx 1.02\times10^{3} E_0, \quad \hbar\omega_\mathrm{b}[\mathrm{keV}] \approx 4.21\times10^{3} E_0\ ,\\
&\nu_\mathrm{peak}[10^{18}\ \mathrm{Hz}] \approx 0.520, \quad \hbar\omega_\mathrm{peak}[\mathrm{keV}] \approx 2.15\ .
\end{aligned} \tag{4.20}$$

In the tables of Section 5, we will list ν rather than the circular frequencies used throughout the paper, and the corresponding energies $\hbar\omega$ will be denoted by ε_b, ε_c, and $\varepsilon_\mathrm{peak}$. The electromagnetic power and the photonic number count, cf. (4.16), scale as

$$\begin{aligned}
&P^{\mathrm{ph}}[\mathrm{GeV\ s^{-1}}] \approx 4.22\times10^{3} E_0^4 R_0^{-2} \approx 3.79 E_0^2 B_0^2\ ,\\
&N^{\mathrm{ph}}[\mathrm{s^{-1}}] \approx 6.18\times10^{9} E_0 R_0^{-1} \approx 1.85\times10^{8} B_0\ .
\end{aligned} \tag{4.21}$$

Wave length, energy and speed (group velocity) of a tachyon relate as [17],

$$\lambda^{\mathrm{tach}} = \frac{hc}{\sqrt{m_\mathrm{t}^2 c^4 + \hbar^2\omega^2}}, \quad \hbar\omega = \frac{m_\mathrm{t} c^2}{\sqrt{\upsilon_{\mathrm{tach}}^2/c^2 - 1}}\ , \tag{4.22}$$

where $hc \approx 12.40$ keV Å. For instance, $\lambda_\mathrm{b}^{\mathrm{tach}} \sim hc/\varepsilon_\mathrm{b}$, cf. (4.20), to be compared with $\lambda_{\mathrm{Compton}}^{\mathrm{tach}} \approx 5.8$ Å (unreduced) and the photonic $\lambda_\mathrm{c}^{\mathrm{ph}}[\text{Å}] \approx 12.4/\varepsilon_\mathrm{c}[\mathrm{keV}]$. As for the tachyonic velocity, we find $\upsilon_{\rm tach}/c - 1 \sim m_{\rm t}^2 c^4/(2\varepsilon_{\rm b}^2)$ at the break energy, and a modest $\upsilon_{\rm tach}/c \approx 1.41$ at the spectral peak. Numerical examples will be given in the tables.

In Sections 5 and 6, we will need to know how a non-singular electron distribution affects the spectral densities (3.22). To this end, we average them with an electronic power-law distribution, $dn(\gamma) \propto \gamma^{-s}\,d\gamma$, of index $s > 1$; the electronic Lorentz factors range in a finite interval, $\gamma_1 \leqslant \gamma \leqslant \gamma_2$, the source count reads $N_{1,2} = \int_{\gamma_1}^{\gamma_2} dn(\gamma)$. In the subsequent sections, we will piece together multi-band spectra with broken power-laws. A synchrotron model of γ-ray burst spectra with an analytic cross-over between power-laws is studied in Refs. [18,19]. A cross-over from power-law to exponentially damped power-law seems to apply to the pervading electron density in the Coma Cluster, derived on the basis of synchrotron and inverse Compton models [20]. Here, we will content ourselves with broken power-laws; the introduction of curvature always involves some arbitrariness in the analytic shape of $n'(\gamma)$, and tends to make integrations rather clumsy. Like the cosmic ray spectrum, the wideband spectra of supernova remnants can be assembled quite convincingly with broken power-laws, cf. Section 6; there is very little evidence for curvature. The TeV flare spectra of the BL Lac object Mrk 501 are genuinely curved [21], suggesting an exponential cutoff factor in $dn(\gamma)$, possibly with a further power-law in the exponential [22]. The averaged linear density (4.23) including an exponential cutoff is still analytically tractable with incomplete Γ-functions, but we would not consider blazars here, and broken power-laws will do for Jupiter's aurora and synchrotron nebulae. We also restrict ourselves to the linear densities (3.23); the leading order of the curvature radiation drops out in the averaging procedure, cf. after (4.24). These densities are generated by ultra-relativistic particles, so that $\gamma_1 \gg 1$. The averaging is carried out via

$$\langle p^{\rm T,lin}(\omega)\rangle_s = \int_{\gamma_1}^{\gamma_2} p^{\rm T,lin}(\omega,\gamma)\Theta(m_{\rm t}c\gamma - \omega)\,dn(\gamma)\,, \tag{4.23}$$

and analogously for $\langle p^{\rm L,lin}(\omega)\rangle_s$. The densities $p^{\rm T,lin}$ and $p^{\rm L,lin}$ are cut off at $\omega_{\rm b} = m_{\rm t}c\gamma$, which is accounted for by the step function Θ in (4.23), cf. the discussion after (3.24). The longitudinal density $p^{\rm L,lin}$ depends on the Lorentz factor only via the cutoff frequency $\omega_{\rm b}$. We thus find the averages,

$$\langle p^{\rm L,lin}(\omega)\rangle_s = N_{1,2}\frac{q^2}{4\pi}\frac{m_{\rm t}^2 c\omega}{\omega^2 + m_{\rm t}^2c^2}\frac{1-(\omega_1/\omega_2)^{s-1}}{1-(\gamma_1/\gamma_2)^{s-1}}(\omega_1/\omega_{\rm b1})^{1-s}\,,$$

$$\omega_1(\omega) := \max(\omega,\omega_{\rm b1}),\quad \omega_2(\omega) := \max(\omega,\omega_{\rm b2}),\quad \omega_{\rm b1,2} := m_{\rm t}c\gamma_{1,2}\,,$$

$$\langle p^{\rm T,lin}(\omega)\rangle_s = \langle p^{\rm L,lin}(\omega)\rangle_s\left(1 - \frac{s-1}{s+1}\frac{\omega^2}{\omega_1^2}\frac{1-(\omega_1/\omega_2)^{s+1}}{1-(\omega_1/\omega_2)^{s-1}}\right)\,. \tag{4.24}$$

Clearly, these densities vanish for $\omega > \omega_{\rm b2}$, where $\omega_1/\omega_2 = 1$. In the high-frequency band, $\omega_{\rm b1} \ll \omega \ll \omega_{\rm b2}$, we find $\langle p^{\rm T,L,lin}\rangle_s \propto \omega^{-s}$, so that the tachyonic spectral index coincides with the electronic power-law index, in strong contrast to the photon index, cf. (4.26). In the low-frequency regime, we recover $\langle p^{\rm T,L,lin}\rangle_s \propto \omega$ for $\omega \ll m_{\rm t}c$, the linear frequency scaling of the distributions (3.23), and $\langle p^{\rm T,L,lin}\rangle_s \propto \omega^{-1}$ holds for $m_{\rm t}c \ll \omega \ll \omega_{\rm b1}$. The curvature radiation generated by the densities (3.16), (3.18) and (3.24) can be averaged like in (4.23), the procedure outlined in (A.23)–(A.31) also applies here, in particular the same variable transformations can be used, with ω replaced by $\omega_{\rm b}$ in (A.25). In leading order (large κ), the high- and low-frequency components cancel each other when integrated with $\omega_{\rm b}^{-s}\,{\rm d}\omega_{\rm b}$, and the higher orders are overpowered by the averaged linear densities (4.24) and do not affect their slopes. The same happens with the integral curvature radiation stemming from a singular electron distribution, cf. (4.4), (4.7) and (4.14).

We integrate the averaged densities (4.24) by making use of $\gamma_1 \gg 1$, to find the averaged transversal power,

$$\langle P^{\rm T}\rangle_s \approx \int_0^\infty \langle p^{\rm T,lin}(\omega)\rangle_s\,{\rm d}\omega$$
$$\sim N_{1,2}\alpha_{\rm q}\frac{m_{\rm t}^2c^4}{\hbar}\left(\log\gamma_1 + \frac{(\gamma_1/\gamma_2)^{s-1}}{1-(\gamma_1/\gamma_2)^{s-1}}\log\frac{\gamma_1}{\gamma_2} + \frac{1}{s-1} - \frac{1}{2}\right)\,, \tag{4.25}$$

and the same for $\langle P^{\rm L}\rangle_s$, with $-1/2$ in the parenthesis dropped. (We have again restored the units, $m_{\rm t} \to m_{\rm t}c/\hbar$). This can also be derived by directly averaging the powers (4.5) and (4.8) with ${\rm d}n(\gamma)$. The linear count rates do not depend in leading order on the Lorentz factor, cf. (4.10), and are therefore not affected by the averaging, apart from the factor $N_{1,2}$, of course. The tachyonic mean frequencies are thus $\langle\omega^{\rm T,L}\rangle_s \sim (2/\pi)\omega_{\rm peak}(\log\gamma_1 + \cdots)$, with the parenthesis completed as in (4.25). This is to be compared with the mean frequencies (4.15) stemming from a singular electron distribution.

For comparison, we quote the electromagnetic average [11],

$$\langle p^{\rm ph}_{\|,\perp}(\omega)\rangle_s = \int_{\gamma_1}^{\gamma_2} p^{\rm ph}_{\|,\perp}(\omega,\gamma)\,{\rm d}n(\gamma) \sim \frac{2^{(s-1)/2}}{8\sqrt{3}\pi}\left(\frac{3s+7}{3s+3} \mp 1\right)$$
$$\times(s-1)\Gamma\left(\frac{3s-1}{12}\right)\Gamma\left(\frac{3s+7}{12}\right)$$
$$\times N_{1,2}\alpha_{\rm e}\hbar\omega_{\rm c1}\frac{1}{\gamma_1^2}\frac{(\omega/\omega_{\rm c1})^{(1-s)/2}}{1-(\gamma_1/\gamma_2)^{s-1}}\,, \tag{4.26}$$

with $p^{\rm ph}_{\|,\perp}(\omega,\gamma)$ defined in (3.19) and the critical photon frequencies $\omega_{\rm c1,2}$ taken at $\gamma_{1,2}$. (The critical frequency scales as $\omega_{\rm c} \propto \gamma^2$, since $\omega_{\rm B} \propto \gamma^{-1}$, cf. (4.1) and (4.2).) This is derived by extending the integration boundaries in (4.26) to zero and infinity and applying (A.30). The photon distribution is strongly peaked, more or less at the mean frequency, cf. (4.16) and after (4.2). The tachyon densities $p^{\rm T,L,lin}$ are much more extended, with a mean frequency largely deviating from the spectral peak at 2.15 keV, cf. (4.15) and Table 2 in Section 5. The extension of the integration boundaries in (4.26) is only permissible if ω lies well within the band $\omega_{\rm c1} \ll \omega \ll \omega_{\rm c2}$. (If the integration boundary in the integrals (A.30) is cut, they admit antiderivatives in terms of Lommel functions with simple asymptotic expansions.) It is only in this band that the averaged photon density has a power-law decay $\propto \omega^{-\alpha}$, with $\alpha := (s-1)/2$. As $\kappa = \omega_{\rm b}/\omega_{\rm c}$ is large throughout the averaging interval, this band noticeably differs from the tachyonic high-frequency band $\omega_{\rm b1} \ll \omega \ll \omega_{\rm b2}$, where the tachyon densities (4.24) admit a power-law slope of index s, even though these bands can overlap.

5. Tachyonic X-rays from Jupiter's magnetosphere

A theory of tachyons should provide clues as to where to find them. In Tables 1–4, we give some estimates to that effect by comparing tachyonic with electromagnetic synchrotron radiation. The examples cover a wide range of electron energies and gyroradii, and a smaller range of magnetic fields. Extremely high magnetic field strengths require different asymptotics, cf. after (4.2), let alone quantum corrections which can completely change the shape of the classical spectral densities by generating a cutoff before the classical spectral peak is reached [11,23]. Even for moderate magnetic field strengths, there is a pronounced tachyonic quantum effect emerging in the low-energy cyclotron limit, unparalleled in electromagnetic theory. There is a threshold velocity, a lower bound on the speed of the (always subluminal) source, below which tachyons can only be radiated as curvature radiation [6]. This threshold velocity is without counterpart in the classical radiation theory, and it coincides, most remarkably, with the speed

Table 1
Entries as defined in Section 4: electron energy E (input), electronic Lorentz factor γ, magnetic field B (input), gyroradius R, critical photon energy ε_c, tachyonic break energy ε_b, tachyonic spectral maximum $\varepsilon_{peak} \approx 2.15$ keV. Magnetic field strengths: 100 and 300 μG for the Crab, 1 and 3 mG for Cas A, 10 and 30 μG for SN 1006. The parameters of SN 1006 apply to RX J1713.7–3946 as well, and the Crab entries E, γ and ε_b also hold for the other remnants. References for the broadband spectra, from soft X-rays to TeV γ-rays, are cited in the text. The wave lengths at 1.4 Jovian radii refer to the photonic spectral peak at $\lambda_c^{ph}/0.286$, cf. Table 2; the respective radio fluxes [28–32] are quoted in the text. The magnetic field strengths are estimates derived in Refs. [34–36,42,44,51,55,56]. The URLs (a–d) point to the technical data sheets of the storage rings

	E(GeV)	γ	B(G)	R(m)	ε_c (keV)	ε_b (keV)	Refs.
Storage rings							
Surf III (NIST)	0.1	196	3.985×10^3	0.837	2.65×10^{-3}	420	a
	0.3	587	1.195×10^4	0.837	0.072	1.3×10^3	
Aladdin (SRC)	0.8	1570	1.28×10^4	2.08	0.55	3.4×10^3	b
SPring-8 (JASRI)	8.0	1.57×10^4	6.79×10^3	39.3	29	3.4×10^4	c
Petra II (HASYLAB)	12	2.35×10^4	2.09×10^3	192	20	5.05×10^4	d
Jupiter's magnetosphere							[34–36]
90 cm @ 1.4 R_J	7.8×10^{-3}	15.3	1.2	217	4.85×10^{-9}	33	[28]
21 cm	0.016	31.3	1.2	445	2.0×10^{-8}	67	[29]
11 cm	0.022	43.05	1.2	612	3.9×10^{-8}	93	[30]
6 cm	0.03	58.7	1.2	834	7.2×10^{-8}	130	[30,31]
2.2 cm	0.05	97.85	1.2	1390	2.0×10^{-7}	210	[32]
Supernova remnants							
Crab Nebula	10^k	$1.96 \times 10^{3+k}$	10^{-4}	$3.3 \times 10^{8+k}$	$6.65 \times 10^{2k-9}$	$4.2 \times 10^{3+k}$	[42,44,51]
			3×10^{-4}	$1.1 \times 10^{8+k}$	$2.0 \times 10^{2k-8}$		
Cassiopeia A			10^{-3}	$3.3 \times 10^{7+k}$	$6.65 \times 10^{2k-8}$		[55]
			3×10^{-3}	$1.1 \times 10^{7+k}$	$2.0 \times 10^{2k-7}$		
SN 1006 and			10^{-5}	$3.3 \times 10^{9+k}$	$6.65 \times 10^{2k-10}$		[56]
RX J1713.7–3946			3×10^{-5}	$1.1 \times 10^{9+k}$	$2.0 \times 10^{2k-9}$		

[a] http://physics.nist.gov/MajResFac/SURF/SURF/accjavan.html
[b] http://www.src.wisc.edu/facilities/Aladdin/parameters.html
[c] http://www.spring8.or.jp
[d] http://www-hasylab.desy.de

Table 2
Entries as defined in Section 4: tachyonic mean energy $\langle\varepsilon^{T,L}\rangle$, tachyonic mean frequency $\langle\nu^{T,L}\rangle$ (transversally and longitudinally radiated), critical photon frequency ν_c, tachyonic break frequency ν_b. Crab entries apply unless listed otherwise. The continuous parameter k labels the electron energy, cf. Table 1, one may envisage $0 \leqslant k \leqslant 7$ as typical range, cf. Section 6. The tachyonic peak frequency, $\nu_{peak}(10^{18}\ \mathrm{Hz}) \approx 0.52$, is independent of the electron energy

	$\langle\varepsilon^T\rangle$ (keV)	$\langle\varepsilon^L\rangle$ (keV)	ν_c (10^{18} Hz)	ν_b (10^{18} Hz)	$\langle\nu^T\rangle$ (10^{18} Hz)	$\langle\nu^L\rangle$ (10^{18} Hz)
SRs						
Surf III	6.5	7.2	6.4×10^{-4}	100	1.6	1.75
	8.0	8.7	0.017	310	1.95	2.1
Aladdin	9.4	10.1	0.13	820	2.3	2.4
SPring-8	12.5	13.2	7.0	8200	3.0	3.2
Petra II	13.1	13.8	4.8	1.2×10^4	3.2	3.3
Jupiter						
	3.05	3.7	1.2×10^{-9}	8.0	0.74	0.90
	4.0	4.7	4.95×10^{-9}	16	0.97	1.1
	4.5	5.15	9.35×10^{-9}	22	1.1	1.2
	4.9	5.6	1.7×10^{-8}	31	1.2	1.35
	5.6	6.3	4.8×10^{-8}	51	1.35	1.5
SNRs						
Crab	$3.15k + 9.7$	$3.15k + 10.4$	$1.6 \times 10^{-9+2k}$	$1.0 \times 10^{3+k}$	$0.76k + 2.3$	$0.76k + 2.5$
			$4.8 \times 10^{-9+2k}$			
Cas A			$1.6 \times 10^{-8+2k}$			
			$4.8 \times 10^{-8+2k}$			
SN & RX			$1.6 \times 10^{-10+2k}$			
			$4.8 \times 10^{-10+2k}$			

of the Galaxy in the microwave background, $\upsilon_{LG}/c \approx m_t/(2m) \approx 2.10 \times 10^{-3}$. Clearly, the ultra-relativistic synchrotron radiation discussed here is not affected by this.

A first orientation as to what one can expect is provided by the storage rings listed in the tables. The frequency peak of the photonic energy density (3.19) is located at $0.286\varepsilon_c$, very close to the photonic mean energy at $0.308\varepsilon_c$, cf. (4.17). This is in contrast to tachyon radiation, where the peak energy of the densities (3.23) at 2.15 keV is clearly separated from the mean frequencies $\langle\varepsilon^{T,L}\rangle$ in Table 2. The photon density is sharply peaked with exponential decay toward higher frequencies, whereas the linear tachyon densities decay as $p^{T,L,\mathrm{lin}} \propto \omega^{-1}$ between 2.15 keV and the break energy in the low MeV region. The tachyonic power radiated as curvature radiation is negligible compared to the linear power $P^{T,L,\mathrm{lin}}$, cf. Table 3; this applies to all examples given in the tables, since the expansion parameter ω_b/ω_c (the ratio of break and critical frequency) is always very large, cf. (4.2). In pulsar magnetospheres, however, the opposite limit is realized, and then the curvature radiation dominates; this will be studied elsewhere. In the ultra-relativistic limit, the power of the linear tachyon radiation varies only weakly with the electronic Lorentz factor, and it can easily surpass the photonic power radiated, in Jupiter's case by a staggering factor of 10^9. The tachyonic

Table 3
The power radiated, cf. Section 4: unpolarized electromagnetic radiation $P^{\rm ph}$, transversal and longitudinal tachyon radiation $P^{\rm T,L,lin}$ (residual, in the limit of infinite curvature radius), tachyonic curvature radiation $P^{\rm T,L,curv}$. Input parameters as in Table 1

	$P^{\rm ph}$ (GeV s^{-1})	$P^{\rm T,lin}$ (GeV s^{-1})	$P^{\rm L,lin}$ (GeV s^{-1})	$P^{\rm T,curv}$ (eV s^{-1})	$P^{\rm L,curv}$ (eV s^{-1})
SRs					
Surf III	0.60	3.3	3.7	13	2.4×10^{4}
	49	4.1	4.5	240	1.0×10^{5}
Aladdin	390	4.8	5.15	970	2.05×10^{5}
SPring-8	1.1×10^{4}	6.4	6.8	9000	6.25×10^{5}
Petra II	2.4×10^{3}	6.7	7.0	3200	3.7×10^{5}
Jupiter					
	3.3×10^{-10}	1.6	1.9	8.6×10^{-6}	19
	1.4×10^{-9}	2.1	2.4	2.25×10^{-5}	31
	2.6×10^{-9}	2.3	2.6	3.4×10^{-5}	38.5
	4.9×10^{-9}	2.5	2.85	5.2×10^{-5}	47
	1.4×10^{-8}	2.9	3.2	1.0×10^{-4}	67
SNRs					
Crab	$3.8\times10^{-14+2k}$	$1.61k+5.0$	$1.61k+5.3$	$2.0\times10^{-8+4k/3}$	$9.4\times10^{-1+2k/3}$
	$3.4\times10^{-13+2k}$			$8.8\times10^{-8+4k/3}$	$1.95\times10^{2k/3}$
Cas A	$3.8\times10^{-12+2k}$			$4.4\times10^{-7+4k/3}$	$4.3\times10^{2k/3}$
	$3.4\times10^{-11+2k}$			$1.9\times10^{-6+4k/3}$	$9.0\times10^{2k/3}$
SN & RX	$3.8\times10^{-16+2k}$			$9.4\times10^{-10+4k/3}$	$2.0\times10^{-1+2k/3}$
	$3.4\times10^{-15+2k}$			$4.1\times10^{-9+4k/3}$	$4.2\times10^{-1+2k/3}$

break energies of the four storage rings range between 0.4 and 50 MeV, as compared with the tachyonic spectral peak at 2.15 keV; the photonic spectral peaks of SURF III lie in the near IR and UV, Aladdin's peak frequency in the extreme UV, and SPring-8 and Petra II radiate hard X-rays. The tachyonic wavelength at 2.15 keV is 4.1 Å, cf. (4.22), to be compared with the photonic counterpart $\lambda_{\rm c}^{\rm ph}/0.286$ and the tachyonic wavelengths $\lambda_{\rm b}^{\rm tach}$ at the break energies in the soft γ-ray band, cf. Table 4. The photon count $N^{\rm ph}$ of the storage rings is by three to four orders higher than the tachyonic emission rate, $N^{\rm T,L,lin}\approx 5.1\times10^{5}\ {\rm s}^{-1}$, cf. Table 4. This does not seem insurmountable, but it comes on top of the very weak coupling of tachyons to matter, by a factor of $\alpha_{\rm q}/\alpha_{\rm e}\approx 1.4\times10^{-11}$ smaller than the electromagnetic interaction, cf. the beginning of Section 4. This factor shows in ratios of tachyonic and photonic ionization cross sections and induced transition amplitudes [13,15]; in the case of Petra II, only one in 10^{14} ionizations is tachyonic. The chances to detect tachyon radiation in storage rings are therefore rather slim, even though the longitudinal polarization and the tachyon speed at the peak frequency can be used as sieves.

We turn to Jupiter's radiation belts [24–27]. The non-thermal flux at 90 cm (0.33 GHz or $0.286\nu_{\rm c}$, cf. Table 1) is 6.2 Jy [28]. At 21 cm (1.4 GHz), a non-thermal flux of 5.3 Jy is quoted in Ref. [29]. At 11 cm (2.7 GHz) a flux of 4.3 Jy and at 6 cm

Table 4
Count rates, wave lengths, group velocity, cf. Section 4: unpolarized photon count $N^{\rm ph}$, polarized tachyon count $N^{\rm T,L,curv}$ (curvature radiation). The ultra-relativistic count rate for the residual radiation (infinite gyroradius) is $N^{\rm T,L,lin} \approx 5.1 \times 10^5\ {\rm s}^{-1}$. Critical photonic wave length $\lambda_{\rm c}^{\rm ph}$, tachyonic wave length at break energy $\lambda_{\rm b}^{\rm tach}$, group velocity at break energy $\upsilon_{\rm tach}$. To be compared with $\lambda_{\rm peak}^{\rm tach} \approx 4.1$ Å and $\upsilon_{\rm tach}/c \approx 1.4$ at the spectral maximum at 2.15 keV

	$N^{\rm ph}({\rm s}^{-1})$	$N^{\rm T,curv}({\rm s}^{-1})$	$N^{\rm L,curv}({\rm s}^{-1})$	$\lambda_{\rm c}^{\rm ph}$(Å)	$\lambda_{\rm b}^{\rm tach}$(Å)	$\upsilon_{\rm tach}/c - 1$ at $\varepsilon_{\rm b}$
SRs						
Surf III	7.4×10^8	3.0×10^{-5}	0.056	4700	0.029	1.3×10^{-5}
	2.2×10^9	1.9×10^{-4}	0.081	170	9.8×10^{-3}	1.45×10^{-6}
Aladdin	2.4×10^9	2.9×10^{-4}	0.060	23	3.7×10^{-3}	2.0×10^{-7}
SPring-8	1.3×10^9	2.7×10^{-4}	0.018	0.43	3.7×10^{-4}	2.0×10^{-9}
Petra II	3.9×10^8	6.3×10^{-5}	7.4×10^{-3}	0.62	2.45×10^{-4}	9.05×10^{-10}
Jupiter						
	2.2×10^5	2.6×10^{-10}	5.85×10^{-4}	2.55×10^9	0.375	2.1×10^{-3}
		3.3×10^{-10}	4.6×10^{-4}	6.1×10^8	0.18	5.1×10^{-4}
		3.7×10^{-10}	4.15×10^{-4}	3.2×10^8	0.13	2.7×10^{-4}
		4.1×10^{-10}	3.75×10^{-4}	1.7×10^8	0.098	1.5×10^{-4}
		4.9×10^{-10}	3.2×10^{-4}	6.2×10^7	0.059	5.2×10^{-5}
SNRs						
Crab	18.5	$4.8 \times 10^{-15+k/3}$	$2.2 \times 10^{-7-k/3}$	$1.9 \times 10^{9-2k}$	$2.95 \times 10^{-3-k}$	$1.3 \times 10^{-7-2k}$
	55.5	$2.1 \times 10^{-14+k/3}$	$4.6 \times 10^{-7-k/3}$	$6.2 \times 10^{8-2k}$		
Cas A	185	$1.0 \times 10^{-13+k/3}$	$1.0 \times 10^{-6-k/3}$	$1.9 \times 10^{8-2k}$		
	555	$4.5 \times 10^{-13+k/3}$	$2.1 \times 10^{-6-k/3}$	$6.2 \times 10^{7-2k}$		
SN & RX	1.85	$2.2 \times 10^{-16+k/3}$	$4.8 \times 10^{-8-k/3}$	$1.9 \times 10^{10-2k}$		
	5.55	$9.7 \times 10^{-16+k/3}$	$1.0 \times 10^{-7-k/3}$	$6.2 \times 10^{9-2k}$		

(5.0 GHz) a non-thermal flux of 3.5 Jy is quoted in Ref. [30] and revised in Ref. [31] as stated. All these fluxes stem from SL9 pre-impact observations. At 2.2 cm (13.8 GHz), a non-thermal flux of 0.42 Jy was observed with Cassini [32]. The photonic spectral density scales with $\nu^{-\alpha}$ when averaged with an electronic power-law density of index $s = 2\alpha + 1$, cf. after (4.26). In the 90–21 cm range, we thus find the photon index $\alpha \approx 0.1$ (as quotient of log(6.2/5.3) and log(1.4/0.33)), a virtually flat photon spectrum. The averaged tachyonic spectral densities inherit the electron index as pointed out after (4.24), and thus scale with $\nu^{-1.2}$ between the break energies of 33 and 67 keV, cf. Table 1. The tachyonic mean energies (averaged with the electronic power-law and Lorentz factors ranging between 15.3 and 31.3, cf. Table 1) are $\langle \omega^{\rm T} \rangle_{s=1.2} \approx 3.5$ keV and $\langle \omega^{\rm L} \rangle_{s=1.2} \approx 4.2$ keV, as indicated after (4.25). The electron energies range in the 8–16 MeV interval, with power-law index 1.2. Between the spectral maximum at 2.15 keV and the spectral break at 33 keV, the tachyonic energy density scales with ν^{-1}, cf. the discussion after (4.24).

In the 21–11 cm range, we find $\alpha \approx 0.32$, and in the 11–6 cm interval $\alpha \approx 0.33$, so that the tachyonic frequency scaling $\propto \nu^{-1.6}$ applies between the break energies of 67 and 130 keV. The 6–2.2 cm interval admits the slopes $\alpha \approx 2$ and $s \approx 5$, so that the tachyonic spectral densities rapidly decay in the 130–210 keV range, with a steep power-law tantamount to an exponential cutoff. The spectral peak is always located at the tachyon mass, at 2.15 keV, in the ultra-relativistic limit, that is. Hence, tachyon emission from Jupiter's radio electrons should be detectable in hard and high-energy X-rays, throughout the 2–130 keV range. Further SL9 pre-impact fluxes at other wavelengths can be found in Refs. [33,34], to the same effect.

Soft X-rays from Jupiter's aurora have been detected on two occasions. The ROSAT spectral map [37–39] shows three peaks in the number count; the peaks at 0.2 and 0.8 keV are probably due to line emissions from precipitating oxygen and sulfur ions, but there is a third tiny peak located at 2 keV, that has not gained attention so far, coinciding with the peak of the tachyonic spectral densities. This peak also shows in the Einstein map [40]. The ROSAT count rate gets sparse above 0.4 keV, and the Einstein map has only a bandpass of 0.2–3 keV, but the three radiation peaks are clearly discernible in the spectral maps. However, the third peak is too tiny to check for the slope of the differential number count above 2 keV, which should scale with ν^{-2} in hard X-rays up to the spectral break at 30 keV. It is tempting to identify this peak with tachyon radiation from the radio electrons. Detection of longitudinally polarized X-rays could be the crucial test as to whether the third radiation peak is tachyonic; transversal and longitudinal tachyons are radiated in equal rates and with nearly equal power in Jupiter's magnetosphere, cf. Table 3.

6. Tachyonic gamma-rays from the shock-heated plasmas of supernova remnants

We will identify tachyonic spectral slopes in the broadband spectra (soft X-ray to TeV γ-ray) of four galactic remnants. We start by approximating the averaged longitudinal density $\langle p^{\mathrm{L,lin}}(\omega)\rangle_s$ in (4.24) by

$$p^{\mathrm{L}}(E \leqslant E_{\mathrm{b1}}) \propto \frac{E}{E^2 + m_{\mathrm{t}}^2 c^4}, \qquad p^{\mathrm{L}}(E_{\mathrm{b1}} \leqslant E < E_{\mathrm{b2}}) \propto (E/E_{\mathrm{b1}})^{-s} . \tag{6.1}$$

The proportionality constant is $N_{1,2}\alpha_{\mathrm{q}} m_{\mathrm{t}}^2 c^4/\hbar$ in both regimes, $E_{\mathrm{b1,2}}$ denotes the break energies $\hbar\omega_{\mathrm{b1,2}}$, and $\mathrm{d}\langle P^{\mathrm{L}}\rangle_s \approx p^{\mathrm{L}}(E)\,\mathrm{d}E$, cf. (4.25). This holds for $E/E_{\mathrm{b2}} \ll 1$, implying $\gamma_1/\gamma_2 \ll 1$. In the same approximation, we write the transversal density $\langle p^{\mathrm{T,lin}}(\omega)\rangle_s$ in (4.24) as

$$p^{\mathrm{T}}(E \leqslant E_{\mathrm{b1}}) \approx p^{\mathrm{L}}(E)\left(1 - \frac{s-1}{s+1}\,\frac{E^2}{E_{\mathrm{b1}}^2}\right),$$

$$p^{\mathrm{T}}(E_{\mathrm{b1}} \leqslant E < E_{\mathrm{b2}}) \approx \frac{2}{s+1}\, p^{\mathrm{L}}(E) . \tag{6.2}$$

We will consider unpolarized radiation, that is, the density $p(E) = p^{\mathrm{T}} + p^{\mathrm{L}}$. The differential energy flux is thus $S'(E) = p(E)/(4\pi d^2)$, where d is the distance to the remnant. This distance and the proportionality factor in (6.1) will drop out in the subsequent ratios. The differential number count, $N'(E)$, relates to the energy flux via $\mathrm{d}S(E) = E\,\mathrm{d}N(E)$, of course. The following discussion is based on the rescaled flux density $f(E) := E^2 N'(E)$. We find, by assembling (6.1) and (6.2),

$$\frac{f(E)}{f(E_{b1})} \approx \frac{2}{1 + m_t^2 c^4/E^2} \frac{s+1}{s+3} , \qquad (6.3)$$

provided $m_t c^2 \leqslant E \ll E_{b1}$. We will consider energies much larger than the tachyon mass, so that this ratio is constant for $E \ll E_{b1}$ and scales as $f(E)/f(E_{b1}) \approx (E/E_{b1})^{1-s}$ for $E_{b1} \leqslant E \ll E_{b2}$. The tachyonic energy density $p(E)$ attains its maximum at the tachyon mass, and $f(E)$ stays nearly constant between break energy and maximum. The tachyonic spectral index s coincides with the electron index and typically ranges in $1 < s < 2$, in the X- and γ-ray bands.

TeV γ-rays from the Crab have been detected in the 0.5–50 TeV range. A differential flux of $N'(E) \approx 2.79 \times 10^{-11}(E/1\ \mathrm{TeV})^{-2.59}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ \mathrm{TeV}^{-1}$ in the 1–20 TeV interval is quoted by the HEGRA Collab. [41]. A recent power-law of the Whipple Group with amplitude 3.12×10^{-11} and exponent 2.57 applies in the 0.5–8 TeV range [42]. The CANGAROO count [43] extends from 7 to 50 TeV, with amplitude 2.76×10^{-11} and exponent 2.53. The subsequent calculations are based on $N'_{\mathrm{TeV}}(E) \approx 2.9 \times 10^{-11}(E/1\ \mathrm{TeV})^{-2.6}$ and $E_{b1} \approx 0.6$ TeV. Above this break energy, the tachyonic spectral index $s \approx 1.6$ applies, as defined after (4.24). We write $f_{\mathrm{TeV}}(E) = E^2 N'_{\mathrm{TeV}}(E)$, so that $f_{\mathrm{TeV}}(E_{b1}) \approx 44.5\ \mathrm{eV\ cm}^{-2}\ \mathrm{s}^{-1}$, and infer from (6.3) that $f_{\mathrm{TeV}}(E)$ cuts as a plateau horizontally through the error bars of four EGRET points and one COS-B point located in the 1–30 GeV range [42,44]. This f_{TeV}-plateau also hits a CELESTE point at 60 GeV [45]. The remaining EGRET flux points below 1 GeV define a power-law slope [46], $N'_{\mathrm{MeV}}(E) \approx 9.1 \times 10^{-9}(E/118\ \mathrm{MeV})^{-2.9}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ \mathrm{MeV}^{-1}$. There are some contaminated data points in this fit, it is feasible that the slope is slightly flatter, especially if one includes two COS-B points below 0.5 GeV, cf. Ref. [44], but not by much. Anyway, we will use $s \approx 1.9$ as tachyonic spectral index in this range. The slope $f_{\mathrm{MeV}}(E) = E^2 N'_{\mathrm{MeV}}(E)$ joins the GeV plateau at $E_1 \approx 380$ MeV, where $f_{\mathrm{TeV}}(E_1) = f_{\mathrm{MeV}}(E_1)$. Hence, $f_{\mathrm{TeV}}(E)$ applies above 380 MeV, up to at least 50 TeV. Below 380 MeV, the slope $f_{\mathrm{MeV}}(E)$ intersects a second plateau value defined by the COMPTEL data points, which give $N'_{\mathrm{ctel}}(E) \approx 1.9 \times 10^{-4}(E/2.6\ \mathrm{MeV})^{-2.0}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ \mathrm{MeV}^{-1}$ for the unpulsed flux in the 1–10 MeV range [47,48]. This plateau is reproduced by $f_{\mathrm{MeV}}(E)$, extended below the spectral break at $E_{b1} \approx 11$ MeV according to (6.3). The break energy is obtained by solving $E_{b1}^2 N'_{\mathrm{ctel}}(E_{b1}) \approx 1.18 f_{\mathrm{MeV}}(E_{b1})$; we find $E_{b1}^2 N'_{\mathrm{ctel}}(E_{b1}) \approx 1.3\ \mathrm{keV\ cm}^{-2}\ \mathrm{s}^{-1}$. Finally, most of the keV range is covered by a broken power-law derived from HEAO 1 and earlier observations [49]; $N'_{\mathrm{X}}(E) \approx 9.23(E/1\ \mathrm{keV})^{-2.13}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ \mathrm{keV}^{-1}$ (unpulsed) applies from 17 keV up to the spectral break at 150 keV, and $N'_{\mathrm{keV}}(E) \approx 2.14 \times 10^{-4}(E/150\ \mathrm{keV})^{-2.54}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ \mathrm{keV}^{-1}$ holds above 150 keV. The slope $E^2 N'_{\mathrm{X}}(E)$ can be continued with broken power-laws to soft X-ray energies and into the UV, optical, IR and radio bands as done in Ref. [50]. The high-energy slope $E^2 N'_{\mathrm{keV}}(E)$ hits the COMPTEL plateau at $E \approx 1.7$ MeV.

The spectrum up to 1.7 MeV is photonic, but the COMPTEL plateau from 1.7 to 11 MeV and the subsequent EGRET slope from 11 to 380 MeV is unlikely to be of electromagnetic origin. A power-law slope preceded by a plateau value of $f(E)$ is characteristic for tachyon radiation. For this reason, I also suggest that the EGRET plateau from 380 MeV to 0.6 TeV and the subsequent power-law slope up to 50 TeV is tachyonic. The EGRET points give a clear picture of the GeV γ-ray spectrum, though they have been criticized on the grounds of contamination and large error bars, as they fail to provide evidence for inverse Compton scattering and pion decay, for the downward slope in the GeV region that is, in all remnants investigated. Plots of the multi-band spectrum with the mentioned flux points can be found in Refs. [42,44,45,47,48].

The energy range of the electrons producing the tachyonic spectral slopes and plateaus can be read off from the tachyonic break energies ε_b in Table 1. The tachyonic slope in the 0.6–50 TeV interval corresponds to $5.15 \leqslant k \leqslant 7.1$, from which we infer electron energies between 0.14 and 13 PeV. The electronic power-law index coincides with the tachyonic spectral index of 1.6. The exponent k parametrizes the electron energy in Table 1. This is to be compared to the 'knee' of the cosmic ray spectrum at $k \approx 6.5$ (≈ 3 PeV) [52]. The second tachyonic slope, between 11 and 380 MeV, corresponds to $0.42 \leqslant k \leqslant 1.96$, which translates into electron energies from 2.6 to 91 GeV with index 1.9.

We turn to the remnant Cas A; γ-rays have been detected in the 1–10 TeV range, with an integral flux of 5.8×10^{-13} cm^{-2} s^{-1} and a slope $N'_{\mathrm{TeV}}(E) \propto (E/1\ \mathrm{TeV})^{-2.5}$ (HEGRA Collab. [53]), from which we infer a proportionality constant of 9.0×10^{-13} cm^{-2} s^{-1} TeV^{-1}. At the break energy $E_{b1} \approx 1$ TeV, we thus find $f_{\mathrm{TeV}}(E_{b1}) \approx 0.90$ eV cm^{-2} s^{-1}, which extends as plateau $f_{\mathrm{TeV}}(E) \approx 1.0$ eV cm^{-2} s^{-1} into the GeV and MeV range according to (6.3) (with $s \approx 1.5$). As for the X-ray slopes [54], the interval 1–120 keV is covered by a broken power-law with spectral break at 16 keV. The differential number count $N'_{\mathrm{X}}(E) \approx 0.038(E/1\ \mathrm{keV})^{-1.8}$ cm^{-2} s^{-1} keV^{-1} applies below 16 keV, and above it steepens to $N'_{\mathrm{keV}}(E) \approx 2.6 \times 10^{-4}\ (E/16\ \mathrm{keV})^{-3.0}$ cm^{-2} s^{-1} keV^{-1}. The $E^2 N'_{\mathrm{keV}}(E)$ slope of the photonic high-energy X-ray tail joins the tachyonic $f_{\mathrm{TeV}}(E)$-plateau at $E \approx 1.1$ MeV, provided that the X-ray count $N'_{\mathrm{keV}}(E)$ extends as unbroken power-law to this energy. The tachyon spectrum spreads from 1.1 MeV to 1 TeV with a flat $f_{\mathrm{TeV}}(E)$, and continues above the spectral break at 1 TeV with a slope $f_{\mathrm{TeV}}(E) \approx 0.90(E/1\ \mathrm{TeV})^{-0.5}$ eV cm^{-2} s^{-1}, cf. after (6.3). The energy of the electrons producing this $f_{\mathrm{TeV}}(E)$ can be inferred from the tachyonic break energies ε_b in Table 1, like for the Crab above. The tachyonic 1–10 TeV range corresponds to $5.38 \leqslant k \leqslant 6.38$, resulting in electron energies of 0.24–2.4 PeV, with a power-law index of 1.5.

As for the remnant SN 1006, an integral X-ray flux of 89 eV cm^{-2} s^{-1} in the 0.1–2.0 keV range is quoted in Ref. [56], and the differential number count between 0.1 and 1.85 keV scales as $N'_{\mathrm{SX}}(E) \approx 0.027(E/1\ \mathrm{keV})^{-2.1}$ cm^{-2} s^{-1} keV^{-1}. Above the spectral break at 1.85 keV, up to 17 keV, the flux scales as $N'_{\mathrm{HX}}(E) \approx 0.047(E/1\ \mathrm{keV})^{-3.0}$ cm^{-2} s^{-1} keV^{-1}, cf. Refs. [56,57]. The observed γ-ray spectrum extends to about 20 TeV (CANGAROO Collab. [58,59]), the spectral index is as yet not well determined. $f_{\mathrm{TeV}}(E)$ is nearly flat between 1 and 5 TeV, indicating a spectral index $s \approx 1$, so that in this range $N'_{\mathrm{TeV}}(E) \approx 9 \times 10^{-12}(E/1\ \mathrm{TeV})^{-2}$ cm^{-2} s^{-1} TeV^{-1}. We thus infer $f_{\mathrm{TeV}}(E) \approx 9$ eV cm^{-2} s^{-1} below 5 TeV, and this plateau extends down to 1 GeV, cutting through the EGRET upper limits in the 1–10 GeV range; a plot of the broadband spectrum with the EGRET flux can be found in Ref. [59]. The EGRET data indicate a spectral break at 1 GeV and a downward slope ranging from about 50 MeV to 1 GeV.

Below 50 MeV, down to high-energy X-rays, there are no flux data available yet. In any case, the $f_{\rm TeV}$-plateau in the GeV range followed by a spectral break in the low TeV region is also present in this remnant, like in Cas A and the Crab, though the evidence is scantier.

The fourth remnant detected in TeV γ-rays is RX J1713.7–3946; the CANGAROO differential flux is $N'_{\rm TeV}(E) \approx 1.6\times10^{-11}(E/1{\rm TeV})^{-2.8}$ cm^{-2} s^{-1} TeV^{-1} in the 0.4–8 TeV range [60]. We find $f_{\rm TeV}(E_{\rm b1}) \approx 19$ eV cm^{-2} s^{-1} at the break energy of 0.8 TeV, followed by the plateau $f_{\rm TeV}(E) \approx 22$ eV cm^{-2} s^{-1} in the GeV region, cf. (6.3) (with $s \approx 1.8$). This plateau cuts horizontally through the data points of the unidentified γ-ray source 3EG J1714–3857 located in the range 0.5–10 GeV. The EGRET spectrum is flat in this region, but starts to steepen below 500 MeV indicating a spectral break, cf. the Crab. A plot of the EGRET points in the wideband spectrum can be found in Ref. [61]. The spectrum from 10 MeV down to hard X-rays has not been measured yet. In the 0.5–10 keV interval, the photonic energy flux scales as $EN'_{\rm X}(E) \propto (E/1\ {\rm keV})^{-1.4}$, and the integral flux in this range is 0.10 keV cm^{-2} s^{-1}, so that $E^2N'_{\rm X}(E) \approx 0.043(E/1\ {\rm keV})^{-0.4}$ keV cm^{-2} s^{-1}, as inferred from ASCA & ROSAT data [62,63]. RXTE observations indicate a steepening of this slope in the 10–30 keV range [64]. The RXTE count does not smoothly join the ROSAT count, there is a discontinuity at 2 keV extending over one order in the count rate. This could be due to background radiation from the Galactic Ridge, but there can also be a tachyonic $f(E)$-plateau extending down to the tachyon mass at 2.15 keV, cf. (6.3), which is not to be subtracted as background radiation.

Tachyonic γ-rays in the 0.8–8 TeV range correspond to electron energies from 0.2 to 2 PeV, parametrized by $5.3 \leqslant k \leqslant 6.3$ in Table 1. The $f_{\rm TeV}$-plateau in the GeV range cutting through the 3EG J1714–3857 flux points again suggests that the GeV and TeV radiation is tachyonic; such an extended plateau is unlikely to be produced by inverse Compton scattering or π^0 decay. The TeV γ-ray flux of this remnant is quite on the Crab scale, whereas the X-ray flux is much weaker and comparable with Cas A, whose TeV radiation is by one order weaker. TeV γ-rays from other remnants such as IC 443 and Tycho have escaped detection. This suggests that there are two energetically different electron populations in remnants with TeV spectra, one emitting tachyonic MeV to TeV γ-rays, the other generating the photonic synchrotron radiation in the radio-to-X-ray bands. I also note that the tachyonic spectral densities in (6.1) and (6.2) stem from the linear densities (4.24), in particular the rescaled flux density $f(E)$, key to the preceding discussion, is independent of the magnetic field; the tachyonic curvature radiation is negligible for the low magnetic field strengths in the remnants, cf. Tables 3 and 4.

The tachyonic luminosity and magnetospheric electron population of γ-ray pulsars can be inferred from COMPTEL and EGRET observations. Due to the strong surface fields, the electromagnetic synchrotron radiation in the γ-ray band is suppressed by a quantum cutoff, but not so tachyonic γ-radiation. This provides an exceptional opportunity to search for tachyon radiation, unspoiled by electromagnetic emission. However, the asymptotics developed here, $\omega_{\rm c}/\omega_{\rm b} \ll 1$, cf. (4.2) and Table 1, does not apply to excessive magnetic field strengths, the opposite limit is realized in the surface fields of γ-ray pulsars.

Very promising sources of superluminal γ-rays are TeV blazars; multi-band spectra of Mrk 421 and Mrk 501 are compiled in Refs. [21,65–68]. The EGRET flux points, however, do not stem from the specific flares for which the broadband spectra are assembled, and there are also rather large variations in the amplitudes and exponents of the TeV γ-ray fluxes, let alone spectral curvature and cutoff energies. The tachyonic interpretation of the γ-ray spectra again hinges on the GeV radiation, which is not known for the individual flares, although the EGRET points in the spectral maps are quite suggestive to that effect. Finally, tachyons do not interact with IR background photons, so that there is no absorption of tachyonic TeV γ-rays from extragalactic sources by electron–positron creation.

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Appendix A. Superluminal spectral and power asymptotics

We start with the stationary phase asymptotics employed in Section 3. The Schott identities (3.7)–(3.9) are a convenient way of rewriting the powers in the multipole expansion (3.4), and the notation customary in synchrotron radiation theory is based on them [8,11]. These identities can be derived by using the power series expansion of J_n^2 and term by term integration; the $J_n'^2$ integration is settled by

$$J_n'^2(nz) = \left[\frac{1}{2n^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\mathrm{d}}{z\,\mathrm{d}z}\right) + 1 - \frac{1}{z^2}\right] J_n^2(nz) \; . \tag{A.1}$$

The asymptotics of electromagnetic synchrotron radiation theory also applies to tachyon radiation. Contrary to the photonic case, however, we have to consider two asymptotic regimes, since x_n/n in (3.1) can be larger or smaller than one. The Bessel functions in (3.7)–(3.9) are replaced by their Nicolson asymptotics, which applies for large positive n and z, so that $z/n \approx 1$, cf. Ref. [69]. If $z > n$,

$$J_n(z) \approx \frac{1}{3}\,(2(z/n-1))^{1/2}(J_{-1/3}(\lambda) + J_{1/3}(\lambda)), \quad \lambda := \frac{n}{3}\,(2(z/n-1))^{3/2} \; , \tag{A.2}$$

$$J_n'(z) \approx \frac{2}{3}\,(z/n-1)(J_{-2/3}(\lambda) - J_{2/3}(\lambda)), \quad J_n''(z) \approx -2(z/n-1)J_n(z) \; , \tag{A.3}$$

$$\int_z^\infty J_n(z)\,\mathrm{d}z \approx \frac{1}{3}\int_\lambda^\infty (J_{-1/3}(\lambda) + J_{1/3}(\lambda))\,\mathrm{d}\lambda \; , \tag{A.4}$$

$$\int_z^\infty z^{-1} J_n(z)\,\mathrm{d}z \approx \frac{1}{n}\int_z^\infty J_n(z)\,\mathrm{d}z - \frac{1}{2n} J_n'(z) \; , \tag{A.5}$$

in leading order of a $1/n$ and $z/n-1$ double series expansion. We also note $\int_0^\infty (1, z^{-1})\times J_n(z)\,dz = (1, n^{-1})$ for $n > -1$ and $n > 0$, respectively, from which the asymptotics of $\int_0^z (1, z^{-1}) J_n(z)\,dz$ can be recovered. Eqs. (A.3) and (A.4) readily follow from (A.2) and the identities

$$(J_{-1/3} + J_{1/3})' = J_{-2/3} - J_{2/3} - (J_{-1/3} + J_{1/3})/(3z) ,$$

$$(J_{-2/3} - J_{2/3})' = -(J_{-1/3} + J_{1/3}) - 2(J_{-2/3} - J_{2/3})/(3z) ,$$

$$J_{-5/3} + J_{5/3} = J_{-1/3} + J_{1/3} + 2(J_{-2/3} - J_{2/3})' . \tag{A.6}$$

In the powers (3.5) and (3.6), we can thus approximate, if $x_n/n \geqslant 1$, cf. (3.1),

$$H^{\mathrm{T}}_{\parallel,\perp}(x_n) \approx \frac{1}{2}\frac{3^{2/3}}{2^{2/3}}\frac{\xi^{2/3}}{n^{2/3}}\left[1 - \frac{1}{3}\int_\xi^\infty (J_{-1/3}(z) + J_{1/3}(z))\,dz + \left(\frac{2}{3} \mp \frac{1}{3}\right)(J_{-2/3}(\xi) - J_{2/3}(\xi))\right] , \tag{A.7}$$

$$H^{\mathrm{L}}(x_n) \approx 1 - \frac{1}{3}\int_\xi^\infty (J_{-1/3}(z) + J_{1/3}(z))\,dz , \tag{A.8}$$

$$\xi := \frac{2n}{3}\,|x_n^2/n^2 - 1|^{3/2} . \tag{A.9}$$

In (A.7), the lower plus-sign refers to the $\perp$-subscript. In this way, we arrive at the spectral densities (3.13) and (3.15) of the low-frequency regime. The relation $F_0 = 2G_0 - L_0$ between the spectral functions (3.14) is obtained by integrating the third identity in (A.6). We note the expansions

$$G_0(\xi) = \frac{2}{3}\frac{\xi^{-2/3}}{2^{1/3}\Gamma(1/3)} - \frac{1}{2}\frac{\xi^{2/3}}{2^{2/3}\Gamma(2/3)} - \frac{1}{2}\frac{\xi^{4/3}}{2^{1/3}\Gamma(1/3)} + \cdots ,$$

$$L_0(\xi) = \frac{2}{3} - \frac{\xi^{2/3}}{2^{2/3}\Gamma(2/3)} - \frac{3}{4}\frac{\xi^{4/3}}{2^{1/3}\Gamma(1/3)} + \cdots , \tag{A.10}$$

where $\Gamma(1/3)\Gamma(2/3) = 2\pi/\sqrt{3}$. In the opposite limit,

$$(F_0, G_0, L_0)(\xi) \sim \sqrt{2/(3\pi\xi)}\cos(\xi + \pi/4) . \tag{A.11}$$

A very different asymptotics applies for $z < n$, familiar from electromagnetic synchrotron radiation theory [11],

$$J_n(z) \approx \frac{1}{\sqrt{3}\pi}(2(1 - z/n))^{1/2} K_{1/3}(\lambda), \quad \lambda := \frac{n}{3}(2(1 - z/n))^{3/2} , \tag{A.12}$$

$$J_n'(z) \approx \frac{1}{\sqrt{3}\pi} 2(1 - z/n) K_{2/3}(\lambda), \quad J_n''(z) \approx 2(1 - z/n) J_n(z) , \tag{A.13}$$

$$\int_0^z J_n(z)\,dz \approx \frac{1}{\sqrt{3}\pi}\int_\lambda^\infty K_{1/3}(\lambda)\,d\lambda , \tag{A.14}$$

$$\int_0^z z^{-1} J_n(z)\,dz \approx \frac{1}{n}\int_0^z J_n(z)\,dz + \frac{1}{2n} J_n'(z) . \tag{A.15}$$

This is again the leading order in a $1/n$ and $1-z/n$ expansion. Eqs. (A.13) and (A.14) follow from (A.12) and the identities

$$K'_{1/3} = -K_{2/3} - (1/(3z))K_{1/3}, \quad K'_{2/3} = -K_{1/3} - (2/(3z))K_{2/3} \ ,$$

$$K_{1/3} = -2K'_{2/3} - K_{5/3}, \quad K_{5/3} = K_{1/3} + (4/(3z))K_{2/3} \ . \tag{A.16}$$

To derive the asymptotics in (A.5) and (A.15), we use the identity $J_n/z = (1/(2n)) \times (J_{n+1} + J_{n-1})$ and write, within the accuracy of (A.2) and (A.12), $J_{n+k}(z) \approx J_n(z(1-k/n))$, so that (A.5) and (A.15) follow by expanding to second order in k/n. One can check by differentiation that these formulas are valid up to terms of $O((z/n-1)^2)$.

Hence, if $x_n/n \leqslant 1$, cf. (3.1), we approximate the powers (3.5) and (3.6) by

$$H^{\mathrm{T}}_{\|,\perp}(x_n) \approx \frac{1}{2\sqrt{3}\pi}(1 - x_n^2/n^2)\left(2K_{2/3}(\xi) - \int_{\xi}^{\infty} K_{1/3}(z)\,\mathrm{d}z \mp K_{2/3}(\xi)\right) \ , \tag{A.17}$$

$$H^{\mathrm{L}}(x_n) \approx \frac{1}{\sqrt{3}\pi}\int_{\xi}^{\infty} K_{1/3}(z)\,\mathrm{d}z \ , \tag{A.18}$$

with ξ as in (A.9). In (A.17), the lower plus-sign again refers to the $\perp$-subscript, and we thus recover the spectral densities (3.16) and (3.18) of the high-frequency regime. The relation $F_\infty = 2G_\infty - L_\infty$ between the spectral functions (3.17) follows by integrating the third identity in (A.16). These functions admit the expansions

$$G_\infty(\xi) = \frac{2}{3}\frac{\xi^{-2/3}}{2^{1/3}\Gamma(1/3)} - \frac{1}{2}\frac{\xi^{2/3}}{2^{2/3}\Gamma(2/3)} + \frac{1}{2}\frac{\xi^{4/3}}{2^{1/3}\Gamma(1/3)} + \cdots \ ,$$

$$L_\infty(\xi) = \frac{1}{3} - \frac{\xi^{2/3}}{2^{2/3}\Gamma(2/3)} + \frac{3}{4}\frac{\xi^{4/3}}{2^{1/3}\Gamma(1/3)} + \cdots \ , \tag{A.19}$$

and rapidly decay for $\xi \to \infty$,

$$(F_\infty, G_\infty, L_\infty)(\xi) \sim (6\pi\xi)^{-1/2}\mathrm{e}^{-\xi} \ . \tag{A.20}$$

The representation of the spectral densities by Airy functions, cf. (3.21), demonstrates analyticity at the break frequency ω_b. To derive this, we write $\xi = (2/3)z^{3/2}$, so that the Airy integral and its (anti-)derivatives read [70,71],

$$\mathrm{Ai}(z) := \frac{1}{\sqrt{3}\pi}\sqrt{z}K_{1/3}(\xi), \quad \mathrm{Ai}(-z) = \frac{1}{3}\sqrt{z}(J_{-1/3}(\xi) + J_{1/3}(\xi)) \ ,$$

$$\mathrm{Ai}'(z) = \frac{-1}{\sqrt{3}\pi} zK_{2/3}(\xi), \quad \mathrm{Ai}'(-z) = -\frac{1}{3}z(J_{-2/3}(\xi) - J_{2/3}(\xi)) \ ,$$

$$\int_0^z \mathrm{Ai}(x)\,\mathrm{d}x = \frac{1}{\sqrt{3}\pi}\int_0^{\xi} K_{1/3}(x)\,\mathrm{d}x,$$

$$\int_0^z \mathrm{Ai}(-x)\,\mathrm{d}x = \frac{1}{3}\int_0^{\xi}(J_{-1/3}(x) + J_{1/3}(x))\,\mathrm{d}x \ , \tag{A.21}$$

all entire functions, and $\mathrm{Ai}''(z) = z\,\mathrm{Ai}(z)$. The integration boundaries may be replaced by $\int_{z,\xi}^{\infty}$, and we note $\int_0^\infty \mathrm{Ai}(\pm x)\,\mathrm{d}x = (1,2)/3$. Accordingly, the spectral functions (3.14) and (3.17) relate to the Airy integral as

$$L_0(\xi) = \int_z^\infty \mathrm{Ai}(-x)\,\mathrm{d}x, \quad G_0(\xi) = -z^{-1}\mathrm{Ai}'(-z)\,,$$
$$L_\infty(\xi) = \int_z^\infty \mathrm{Ai}(x)\,\mathrm{d}x, \quad G_\infty(\xi) = -z^{-1}\,\mathrm{Ai}'(z)\,, \tag{A.22}$$

where $z = (3\xi/2)^{2/3}$. (By the way, $\mathrm{Ai}'(-z)$ is the derivative of $\mathrm{Ai}(z)$ taken at $-z$.) The expansions (A.10) and (A.19) follow from those of the (anti-)derivatives of $\mathrm{Ai}(\pm z)$, listed in Ref. [71]. By substituting (A.22) into the transversal densities (3.13) and (3.16), we easily see that they are the analytic ω-continuation of each other, cf. (3.20) and (3.21). Similarly, the longitudinal densities (3.15) and (3.18) define the same analytic function, determining their high/low-frequency counterpart by analytic continuation.

We turn to the frequency integration of the spectral densities (3.13) and (3.15). The power radiated as curvature radiation in the low-frequency regime, cf. (4.4) and (4.7), and the corresponding number counts can be assembled from the integrals

$$C^\alpha_{F_0,G_0,L_0} := \int_0^{\omega_\mathrm{b}} (\xi^{2/3}F_0(\xi), \xi^{2/3}G_0(\xi), L_0(\xi))\frac{\omega^\alpha \mathrm{d}\omega}{\omega^2 + m_\mathrm{t}^2 c^2}\,. \tag{A.23}$$

The spectral functions are defined in (3.14), with $\xi(\omega)$ and ω_b in (3.11). After rescaling with ω_b, we introduce a new integration variable, $\omega^4 x^2 = (1-\omega^2)^3$, so that

$$C^\alpha_{F_0,G_0,L_0} \sim -\omega_\mathrm{b}^{\alpha-1}\int_0^\infty ((\kappa x)^{2/3}(F_0,G_0)(\kappa x), L_0(\kappa x))\omega^{\alpha-2}(x)\,\mathrm{d}\omega(x)\,, \tag{A.24}$$

up to terms of $O(\gamma^{-2})$. The parameter κ is defined in (3.11); we need the $\kappa \to \infty$ asymptotics of these integrals, cf. the discussion following (4.2). To this end, we expand in (A.24),

$$\omega(x) = 1 \mp (1/2)x^{2/3} + (5/24)x^{4/3} \mp (1/16)x^2 + \cdots\,,$$
$$\omega^\beta\omega' = \mp(1/3)x^{-1/3}(1 \mp (3\beta+5)x^{2/3}/6 + (\beta+1)(\beta+3)x^{4/3}/8 \mp \cdots)\,. \tag{A.25}$$

Only the upper signs are relevant at this point, the lower ones will be used in (A.29). The asymptotic expansion of integrals (A.24) is thus effected by (A.25) combined with the Bessel integrals [69],

$$(F_0^{(\mu)}, G_0^{(\mu)}, L_0^{(\mu)}) := \int_0^\infty (F_0(x), G_0(x), L_0(x))x^\mu\,\mathrm{d}x$$
$$= \frac{2^\mu}{3}\left(\frac{3\mu+5}{3\mu+3}, 1, \frac{3\mu+1}{3\mu+3}\right)$$
$$\times\left(\frac{\Gamma(1/6+\mu/2)}{\Gamma(1/6-\mu/2)} - \frac{\Gamma(5/6+\mu/2)}{\Gamma(5/6-\mu/2)}\right)\,, \tag{A.26}$$

in analytic continuation. Hence,

$$L_0^{(-1/3)} = \frac{1}{2^{1/3}\Gamma(1/3)}, \quad F_0^{(1/3)} = \frac{3}{2}G_0^{(1/3)} = 3L_0^{(1/3)} = \frac{-1}{2^{2/3}\Gamma(2/3)} ,$$

$$3F_0^{(1)}/4 = G_0^{(1)} = 3L_0^{(1)}/2 = -4/9 . \tag{A.27}$$

$L_0^{(-1/3)}$ is calculated from (A.26) by means of an ε-regularizer, $\mu = -1/3 + \varepsilon$. The use of expansion (A.25) over the whole integration range, the interchange of summation and integration, and the analytic continuation in μ can be justified as follows. We split the integration range in (A.24) into two intervals, $[0, \delta]$ and $[\delta, \infty]$, where the cut δ lies within the convergence radius of (A.25). In the lower range, the integrals (A.24) can be reduced, by means of (A.25), to integrals of type $\int_0^\delta J_\alpha(\kappa x)x^\beta\,\mathrm{d}x$, which admit antiderivatives in terms of Lommel functions and thus straightforward asymptotic expansions for $\kappa \to \infty$, cf. Ref. [69]. The integration over $[\delta, \infty]$ is settled by replacing in (A.24) the spectral functions (A.22) by their asymptotic expansions for large argument [71], instead of expanding $\omega(x)$. The resulting Fourier integrals admit a standard asymptotic expansion [72] which entirely hinges on the lower integration boundary δ (where (A.25) still applies), since at the upper end $\omega(x \to \infty) \sim x^{-1/2}$. Collecting terms, we find the expansion indicated in (A.24)–(A.26).

The expansion of the radiant powers in the high-frequency regime, cf. (4.13), is settled analogously. The integration of the transversal and longitudinal spectral densities (3.16) and (3.18) requires integrals of type

$$C^\alpha_{F_\infty,G_\infty,L_\infty} := \int_{\omega_b}^\infty (\xi^{2/3}F_\infty(\xi), \xi^{2/3}G_\infty(\xi), L_\infty(\xi))\frac{\omega^\alpha\,\mathrm{d}\omega}{\omega^2 + m_t^2c^2} , \tag{A.28}$$

with the spectral functions (3.17). We change the integration variable, $\omega^4x^2 = (\omega^2 - 1)^3$, so that

$$C^\alpha_{F_\infty,G_\infty,L_\infty} \sim \omega_b^{\alpha-1}\int_0^\infty ((\kappa x)^{2/3}(F_\infty, G_\infty)(\kappa x), L_\infty(\kappa x))\omega^{\alpha-2}(x)\,\mathrm{d}\omega(x) , \tag{A.29}$$

and substitute expansion (A.25) with the lower signs. The use of this expansion beyond its convergence radius is justified by Watson's Lemma [72], because of the exponential decay of the integrands, cf. (A.20); we have $\omega(x \to \infty) \sim x$ in this case. A partial integration with regard to F_∞ and L_∞ gives

$$(F_\infty^{(\mu)}, G_\infty^{(\mu)}, L_\infty^{(\mu)}) := \int_0^\infty (F_\infty(x), G_\infty(x), L_\infty(x))x^\mu\,\mathrm{d}x$$

$$= \frac{2^{\mu-1}}{\sqrt{3}\pi}\left(\frac{3\mu+5}{3\mu+3}, 1, \frac{3\mu+1}{3\mu+3}\right)\Gamma\left(\frac{\mu}{2}+\frac{1}{6}\right)\Gamma\left(\frac{\mu}{2}+\frac{5}{6}\right) , \tag{A.30}$$

to be compared with (A.27),

$$L_\infty^{(-1/3)} = L_0^{(-1/3)}, \quad (F, G, L)_\infty^{(1/3)} = -(F, G, L)_0^{(1/3)} ,$$

$$3F_\infty^{(1)}/4 = G_\infty^{(1)} = 3L_\infty^{(1)}/2 = 2/9, \quad 3F_\infty^{(0)}/5 = G_\infty^{(0)} = 1/\sqrt{3} . \tag{A.31}$$

$L_{\infty}^{(-1/3)}$ is obtained by ε-expansion of the first Γ-function, otherwise there is no analytic continuation necessary in (A.30). The asymptotic expansion of integrals (A.28) is thus obtained by substituting (A.25) (with lower signs) into (A.29), followed by an interchange of summation and integration, and a subsequent application of (A.30). The curvature radiation, that is, the radiated powers and the number counts discussed in Section 4, can be assembled from integrals (A.23) and (A.28).

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