Relativistic quantum chaos in Robertson–Walker cosmologies

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Open Robertson–Walker cosmologies of multiple spatial connectivity provide a challenging example for the possible influence of the global topological structure of space-time on the laws of microscopic motion. Free geodesic motion is investigated in such cosmologies in the context of first quantization. A unique localized wave field, a solution of the Klein–Gordon equation, is found as a consequence of the topological structure of the spacelike slices $t = \text{const}$ of the manifold. This solution is closely related to the collection of the bounded chaotic trajectories. The link is provided by the quasi-self-similar limit set of the group of covering transformations on the boundary of the universal covering space of the spacelike sections. It is this fractal set from which the covering geodesics of the bounded trajectories emerge, its Hausdorff measure and dimension determine the localized wave field.

I. INTRODUCTION

We consider Robertson–Walker (RW) cosmologies whose spacelike slices $t = \text{const.}$ are three-manifolds of multiple connectivity, infinite volume, and negative sectional curvature. A typical class of such manifolds is that of smooth thickened surfaces: imagine a two-sphere with some handles attached and imagine the material of which this surface is formed as thick. One gets a three-dimensional space with an interior and exterior boundary, topologically $I \times S$, fibering over a finite interval I, the fibers S being compact Riemann surfaces of genus $g \geqslant 2$.

If the boundaries are removed such three-manifolds can be endowed with a metric of constant sectional curvature, that becomes singular at the boundary and gives rise to infinite volume.

The global topological structure of the RW cosmologies we treat here is thus $\mathbb{R}^{(+)} \times I \times S$, where $\mathbb{R}^{(+)}$ is the real (positive) axis. The construction and the geometrical and topological properties of such manifolds will be dealt with in Sec. II.

Geodesic motion in $\mathbb{R}^{(+)} \times I \times S$, which is locally endowed with a RW line element, we will treat (Sec. III) by embedding the manifold into its universal covering space $\mathbb{R}^{(+)} \times B^3$. Here, B^3 denotes hyperbolic space, a shell of the Minkowski hyperboloid, or the Poincaré ball, isometric to it. We will find that there is a special class of trajectories in $\mathbb{R}^{(+)} \times I \times S$, namely those that stay during the whole time evolution in a finite domain that is expanding at the same rate as three-space itself.

These bounded trajectories have covering geodesics in $\mathbb{R}^{(+)} \times B^3$ whose initial or end points lie in the limit set of the group of covering transformations. This limit set constitutes a fractal quasi-self-similar curve on the boundary of B^3. The bounded trajectories are, as in the nonrelativistic case, chaotic,[1] having the Bernoulli property.

On the other hand, with the limit set of the covering group there is also associated a localized, square-integrable wave field, a solution of the Klein–Gordon equation on $\mathbb{R}^{(+)} \times I \times S$. In Sec. IV we give an integral representation of the space part of this wave field in terms of the Hausdorff measure of the limit set, which in the classical case determines the bounded trajectories.

This localized state as well as the existence of bounded chaotic trajectories is an effect of the global topological structure of the space, and is absent in the traditional open RW models of simple spatial connectivity.

In Secs. IV and V we discuss the time evolution of the energy of wave fields, especially in the early and late stages of the cosmic expansion, and in Sec. V we give some examples, the static case, de Sitter space, and finally a space that is flat, with an expansion factor that is linear in time.

II. RW GEOMETRIES OF MULTIPLE CONNECTIVITY AND NEGATIVE SPATIAL CURVATURE

Cosmological line elements complying with the principles of homogeneity and isotropy can be represented as[2–4]

$$ds^2 = -c^2\,dt^2 + a^2(t)\,d\sigma^2, \tag{1}$$

where $d\sigma$ is the line element of a three-dimensional space of constant curvature, which we will assume to be negative. The expansion factor $a(t)$ determines the Gaussian curvature of the spacelike slices $t = \text{const.}$; cf. (5) and (6). The metric $d\sigma^2$ is usually represented as

$$d\sigma^2 = R^2[d\rho^2 + \sinh^2(\rho)(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2)], \tag{2}$$

$0 \leqslant \rho \leqslant \infty$, and has sectional Gaussian curvature $-1/R^2$.

Introducing a new time variable

$$t'(t) = \int_{\text{const.}}^{t} \frac{dt}{a(t)}$$

we can make (1) conformal to a static metric

$$ds^2 = \tilde{a}^2(t')[-c^2\,dt'^2 + d\sigma^2], \tag{3}$$

$\tilde{a}(t') := a(t(t'))$.

In the following we use a form of (2) that is conformal to the line element of Euclidean three-space, the change of coordinates

$$\sinh \rho = \frac{2r/R}{1 - r^2/R^2}, \quad 0 \leqslant r/R \leqslant 1,$$

transforms $a^2(t)d\sigma^2$ of (2) in the metric of the Poincaré ball B^3, $r := |\mathbf{x}| < R$, cf. Refs. 5 and 6,

$$a^2(t)d\sigma^2 = \frac{4a^2(t)d\mathbf{x}^2}{(1 - |\mathbf{x}|^2/R^2)^2}. \tag{4}$$

In B^3 we will construct representations of the three-manifolds that form the spacelike sections of our models.

In the discussion of the wave equation in Sec. IV we need explicit formulae for the curvature scalar of the line elements (1) and (3). Using the sign convention for the curvature tensor as in Refs. 3 and 7, the curvature scalar $\widehat{R}_{\text{space}}$ of the sections $t = \text{const.}$ of (1), (3) reads

$$\widehat{R}_{\text{space}} = 6K = -6/a^2(t)R^2, \tag{5}$$

where $a(t)$ is dimensionless and K the Gaussian curvature of the two-sections of the spacelike slices.

The four-dimensional curvature scalar $\widehat{R}$ of metric (1) or (3) then reads[7,8]

$$\widehat{R} = 6\left[\frac{-1}{a^2(t)R^2} + \frac{\ddot{a}(t)}{c^2a(t)} + \frac{\dot{a}^2(t)}{c^2a^2(t)}\right]$$

$$= 6\left[\frac{-1}{R^2\tilde{a}^2(t')} + \frac{\tilde{a}(t')}{c^2\tilde{a}^3(t')}\right]; \tag{6}$$

the dots denote derivatives with respect to the time variables. If $a(t)$ is a constant, then $\widehat{R}$ reduces to $\widehat{R}_{\text{space}}$. In Sec. V we will encounter an example with $\widehat{R} = 0$, $\widehat{R}_{\text{space}} \neq 0$ ($a(t) = \Lambda t$).

As in Refs. 1, 9, and 10 we model three-space $I \times S$ in the Poincaré ball B^3 as a non-Euclidean polyhedron with face identification (analogous to the modeling of a torus of zero curvature in the Euclidean plane by identifying the sides of a square; see also Ref. 11 for Riemann surfaces). We emphasize that this polyhedron does not change in time, nor does the radius R of the Poincaré ball, if we use the metric (1), (4) on $I \times B^3$. What varies in B^3 is the Gaussian curvature (5) and the geodesic distance between two points, measured via (4).

For the sake of self-containedness we sketch shortly the construction of polyhedra in B^3 that give rise to manifolds $I \times S$.

The faces of the polyhedron lie on totally geodesic planes, i.e., on spherical caps orthogonal to the boundary sphere S_∞ of B^3, where the metric (4) gets singular. Geodesics in B^3 are arcs of circles orthogonal to S_∞.

The identification of the faces is carried out by elements of the invariance group SL(2, $\mathbb{C}$) of the metric (4), for explicit formulae for the group action in B^3 see Refs. 1 and 6. The polyhedron has two open ends, namely faces lying on S_∞, that are not identified, and which constitute the two boundary components of $I \times S$.

Due to these open ends the volume of the polyhedron measured by (4) is infinite, $\mathbb{R}^{(+)} \times I \times S$ belongs therefore to the open models, the spacelike fibers have infinite volume, and B^3 is their universal covering manifold.[12,13] The group Γ of covering transformations is generated by the face-pairing transformations of the polyhedron. This group, applied to the polyhedron, say F, gives a tesselation $\Gamma(F)$ of the covering space B^3 that gets filled with isometric images of F, which cover B^3 completely without overlappings. There are accumulation points of images of F, located on S_∞, constituting a closed fractal Jordan curve $\Lambda(\Gamma)$ on S_∞, the limit set of Γ.

Figures 1(a)–1(d) show tesselations of S_∞, which are induced from the tiling of B^3, and stereographically projected onto the complex plane. The tiling of the second connection component, enclosing the point at infinity, has not been drawn. The large black polygon is an open end of the polyhedron F, one of the boundary surfaces of $I \times S$, if its sides (circular arcs) are properly identified in pairs. The tesselations are obtained by applying Γ to this polygon. The genus of the fibers S is 19 in all four examples; in fact the polygons have many more sides that are sitting on the vertices and are many orders of 10 smaller than the visible ones. There are bounds on the Hausdorff dimensions of such limit sets that can be circumvented only by increasing the genus of the fibers.[14]

Likewise the second boundary surface of $I \times S$ is a polygon containing the point at infinity, Γ applied to it tesselates the remaining part of $\mathbb{C}$ and provides the exterior approximation to $\Lambda(\Gamma)$.

From the tesselation one can read off the Hausdorff dimension δ of $\Lambda(\Gamma)$. It is related to the number of tiles that one needs to get a uniformly accurate approximation[15,16] of the curve by the tiling: $\delta(\text{a}) = 1.386$, $\delta(\text{b}) = 1.393$, $\delta(\text{c}) = 1.381$, $\delta(\text{d}) = 1.452$ (all up to ± 0.003, for the calculation of δ see Ref. 1).

The topological structure does not determine the global metric structure of three-space nor of the four-manifold, locally determined by (1). In fact the four examples in Figs. 1(a)–1(d) correspond to nonisometric manifolds $I \times S$ (there does not exist an isotopic distance-preserving diffeomorphism). Only polyhedra whose covering groups Γ are SL(2,$\mathbb{C}$)-conjugated are globally isometric.

Though the topological structure does not determine the metric structure, it restricts it however: the space of metrics that are locally of the form (4), and that can be carried by a manifold $I \times S$, can be parametrized by $12(g-1)$ independent real parameters, g the genus of S (deformation space, cf. Refs. 17 and 18). The same holds true for $\mathbb{R}^{(+)} \times I \times S$ and the line element (1), with a given $a(t)$.

The Jordan curve $\Lambda(\Gamma)$ of the limit points of the covering group Γ of three-space is crucial in determining the bounded geodesics on the four-manifold. This and the concept of boundedness in a time-dependent metric will be discussed in Sec. III.

Correspondingly, with this curve, its Hausdorff measure, and Hausdorff dimension there is associated a unique localized, square-integrable solution of the Klein–Gordon equation, which has otherwise purely continuous spectrum. The existence of bounded trajectories and localized states is a consequence of the global topological structure, the multiple connectivity of the manifold. This localization phenomenon will be dealt with in Sec. IV.

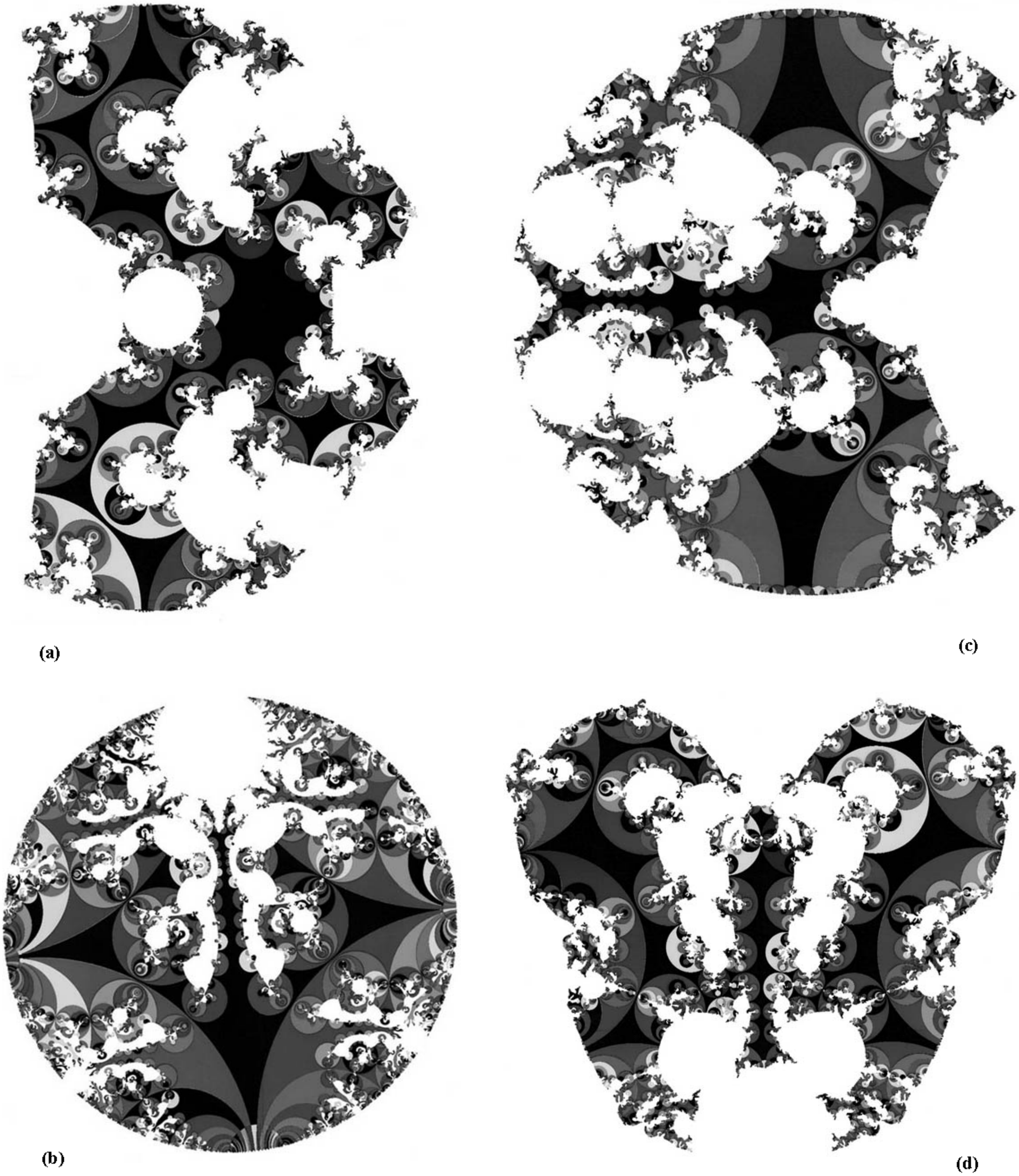
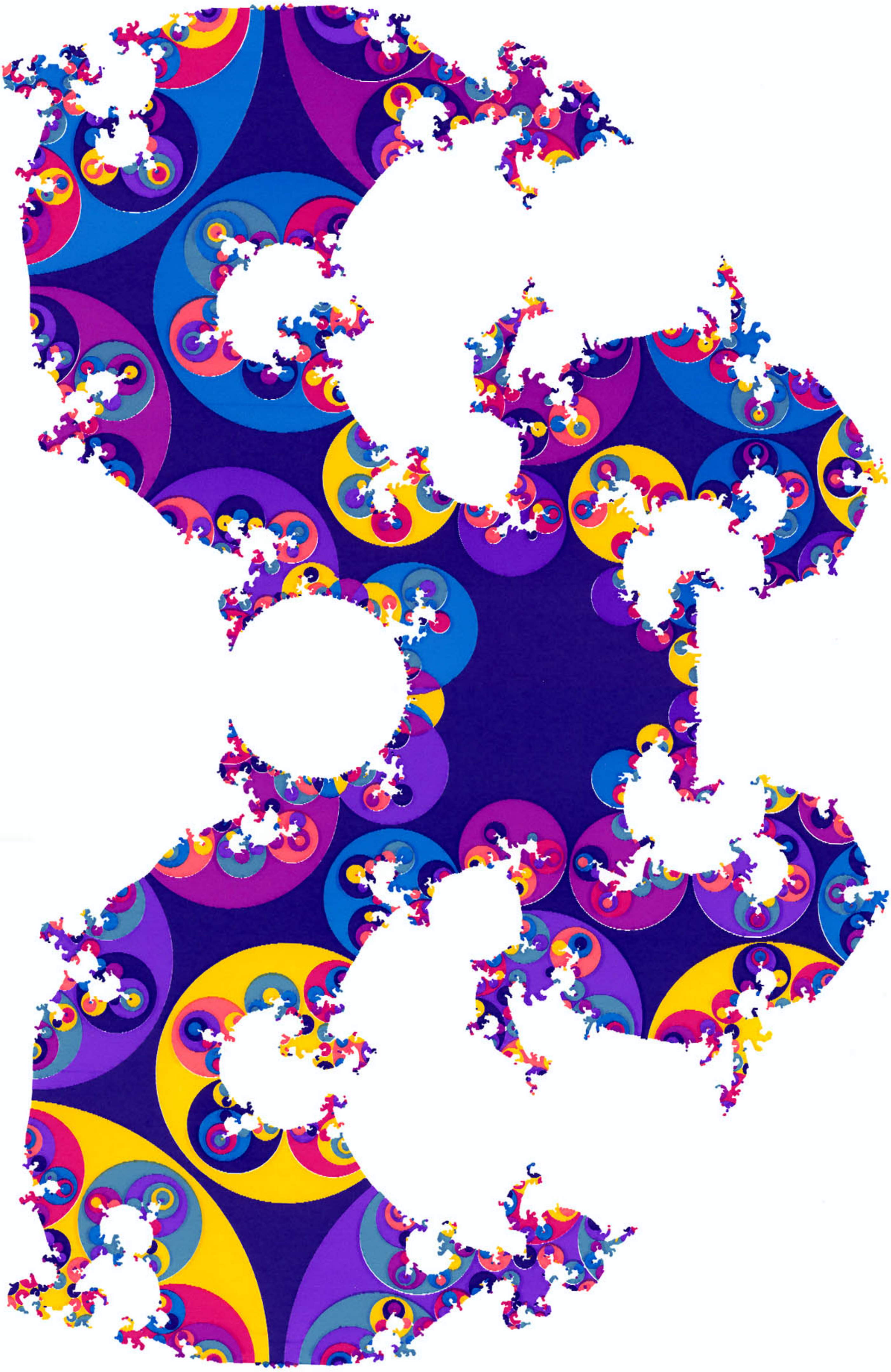
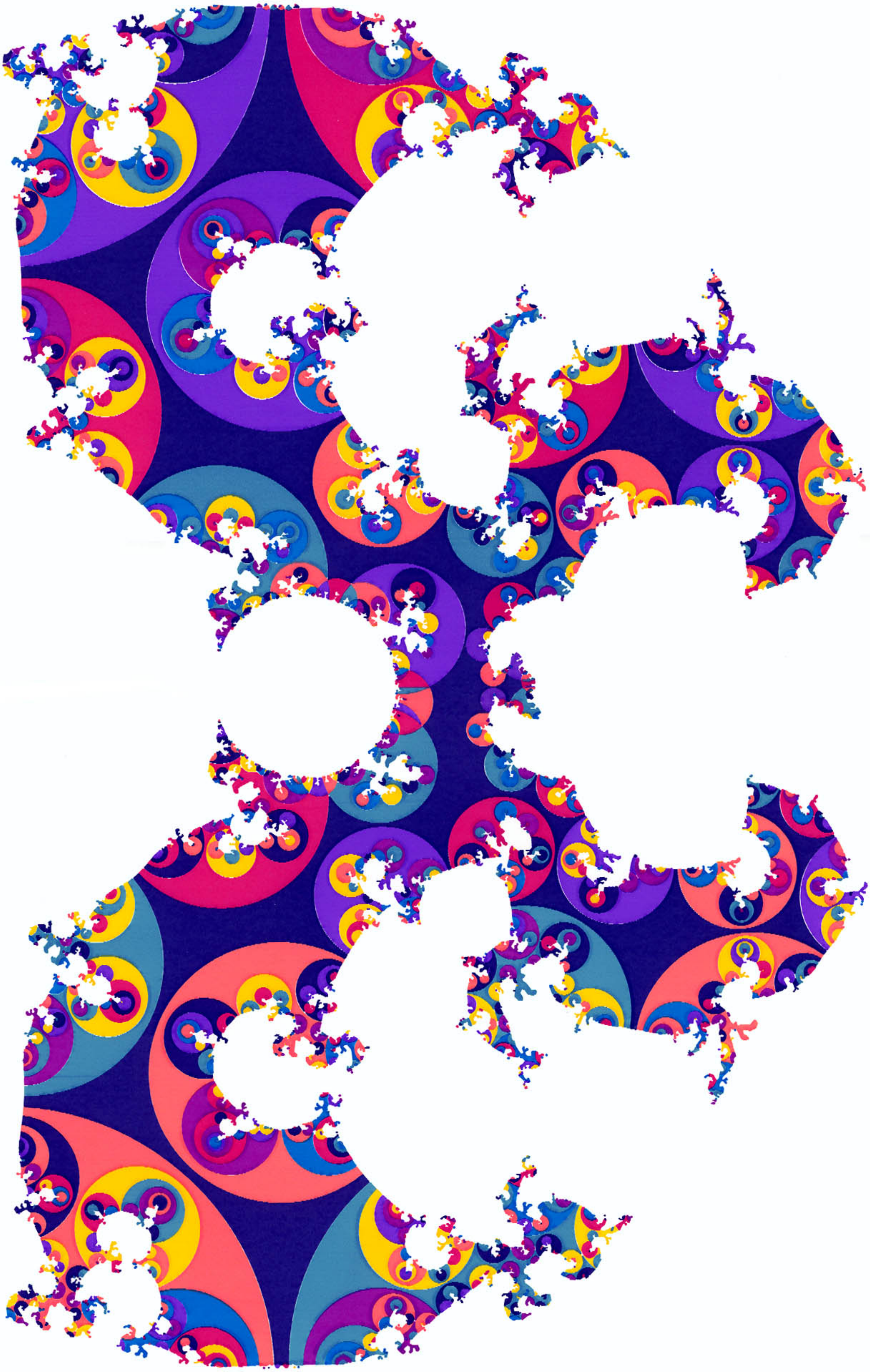


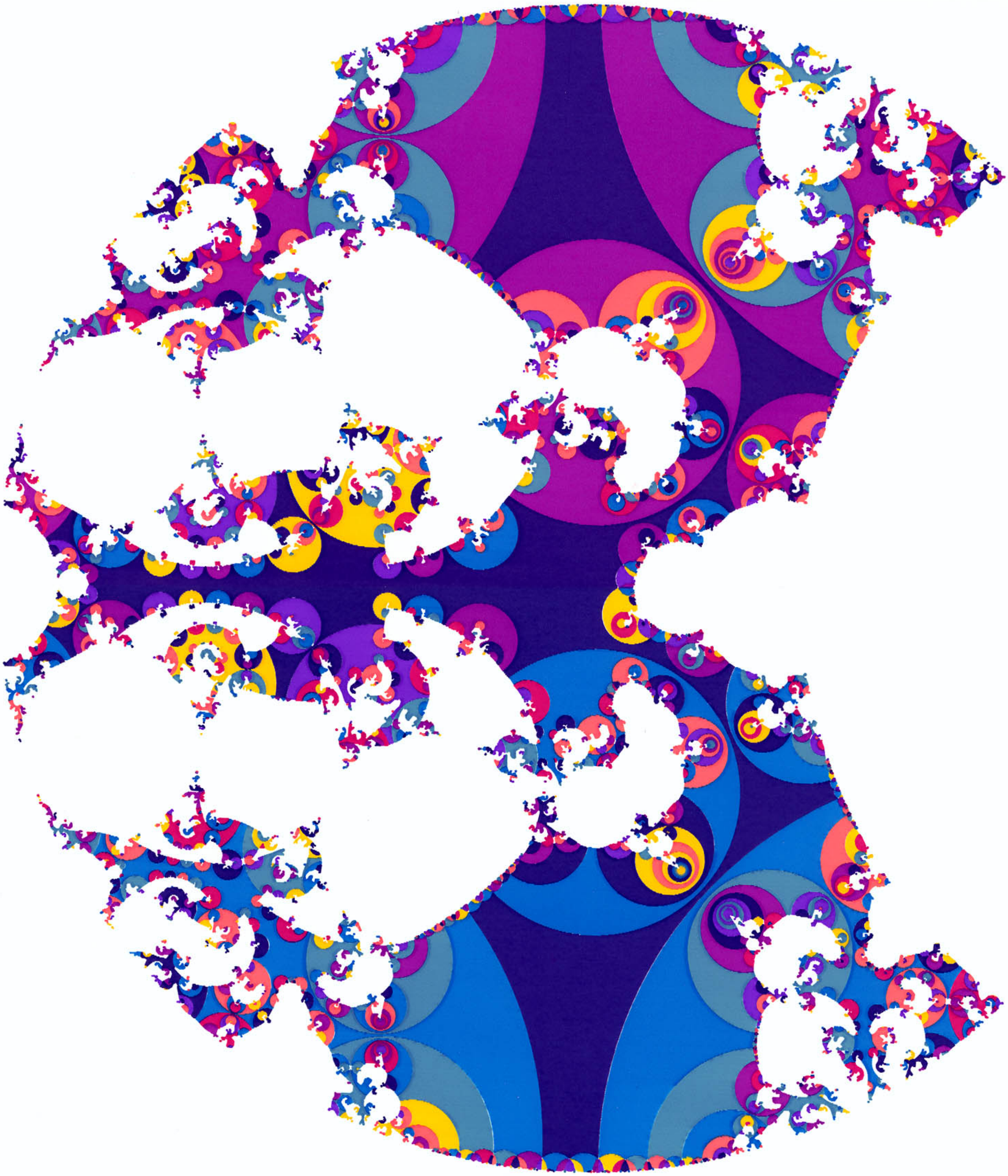
FIG. 1. (a)–(d) Tilings induced on the boundary S_∞ of the universal covering space B^3 of the spacelike slices $I \times S$, $t = \text{const.}$, by the polyhedral tesselation of B^3 (cf. Secs. II and III). The images are stereographic projections of S_∞, the sphere at infinity of the Poincaré ball B^3, onto the complex plane. Only the component of the tiling on the inside of the quasi-self-similar[21] Jordan curve $\Lambda(\Gamma)$, which is homeomorphic to a circle, has been drawn. The large black domain, say f, is one of the boundary surfaces of three-space $I \times S$. The tiling of S_∞ is performed generation by generation by applying the covering group Γ to f. The tiles are the boundary components of isometric images of $I \times S$, which accumulate at $\Lambda(\Gamma)$. From the limit set $\Lambda(\Gamma)$ the covering geodesics of the bounded trajectories emanate, and its Hausdorff measure and dimension δ determine a unique localized wave field of the wave equation on the four-dimensional manifold $\mathbf{R}^{(+)} \times I \times S$. The tilings (a)–(d) correspond to nonisometric manifolds of the same topological structure $I \times S$, I a finite open interval, S a Riemann surface, $g(S) = 19$; $\delta(\text{a}) = 1.386$, $\delta(\text{b}) = 1.393$, $\delta(\text{c}) = 1.381$, $\delta(\text{d}) = 1.452$.



(a)



(b)



(c)

(d)

III. CLASSICAL MECHANICS: THE EMERGENCE OF BOUNDED TRAJECTORIES AS A TOPOLOGICAL EFFECT

We treat at first classical motion in the universal covering space $\mathbb{R}^{(+)}\times B^3$ (cf. Sec. II) of the four-manifold, and realize then the chaotic trajectories in $\mathbb{R}^{(+)}\times I\times S$ by projecting the trajectories of $\mathbb{R}^{(+)}\times B^3$ into it (compare also the nonrelativistic case in Ref. 1).

To treat light rays and trajectories of massive particles on equal footing, it is convenient to start with the line element (3), (4). (We write in the following t for t'.)

The Lagrange function reads then

$$L^2=\tilde{a}^2(t)\,[c^2\dot{t}^2(s)-(4/(1-r^2/R^2)^2)(\dot{r}^2(s)+r^2\dot{\varphi}^2(s))]. \tag{7}$$

We have chosen in (4) polar coordinates r, θ, φ and put $\theta=\pi/2$, $\dot{\theta}=0$ as the initial condition. Motion then takes place on the hyperplane $\theta=\pi/2$.

A first integral is

$$L^2(s)\equiv\lambda>0. \tag{8}$$

For massive particles we choose $\lambda=1$, the parameter s is then the arc length, and in the limit $\lambda\to 0$ we obtain 0-geodesics.

A second integral of L^2 is

$$-\tilde{a}^2(t)r^2\dot{\varphi}(s)/(1-r^2/R^2)^2\equiv M. \tag{9}$$

Finally we have from (7), (8)

$$c^2\frac{d}{ds}(\tilde{a}^2(t)\dot{t})-\lambda\frac{\tilde{a}'(t)}{\tilde{a}(t)}=0, \tag{10}$$

where the prime here denotes derivation with respect to t. If we introduce in (10) t as an independent variable, and define $q:=\dot{t}$, we get a linear differential equation for q^2 with the solution

$$q^2=\lambda c^{-2}\tilde{a}^{-2}(t)+\mu\tilde{a}^{-4}(t), \tag{11}$$

where μ an arbitrary real parameter.

In (8) and (9) we replace $\dot{\varphi}$ and $\dot{r}$ by $q\varphi'$ and qr', i.e., we take time again as an independent parameter. Then to solve (8), (9) in terms of $r(t)$, $\varphi(t)$ we make the ansatz

$$r^2-2mr\cos\varphi+R^2=0, \tag{12}$$

which means to try a trajectory that has the shape of a B^3-geodesic, a circular arc orthogonal to S_∞, centered at $|\mathbf{m}|=m$.

With the help of (12) we eliminate φ' from (8), the resulting equation for $r(t)$ can be immediately integrated. Likewise we eliminate via (12) φ' from (9) and obtain again an equation for $r(t)$ that we integrate. As a consistency condition we get then the value of m in (12) in terms of the integration constants M and μ in (9), (10),

$$\mu=\frac{16m^2M^2}{R^4c^2}\left(1-\frac{R^2}{m^2}\right)\geqslant 0. \tag{13}$$

For $r(t)$ we then have

$$\frac{r^2(t)}{R^2}=\frac{\eta^2+1/4R^4-\eta R^{-2}\sqrt{1-R^2/m^2}}{\eta^2+1/4R^4+\eta R^{-2}\sqrt{1-R^2/m^2}}, \tag{14}$$

with η given by

$$\eta(t'):=\frac{1}{2R^2}\times\exp\left[\pm\frac{c}{R}\int_{\text{const}}^{t'}(1+\lambda\mu^{-1}c^{-2}\tilde{a}^2(t))^{-1/2}dt\right]. \tag{15}$$

[We write again t' for the time variable as in (3).] Since $\eta\geqslant 0$ we have $0\leqslant r(t)/R\leqslant 1$. The $\pm$ sign depends on the sign of M in (9).

In (15) η has been calculated in the context of the line element (3). If we use the line element (1) we get for η in (14)

$$\eta(t):=\frac{1}{2R^2}\exp\left[\pm\frac{c}{R}\int_{\text{const}}^{t}(1+\lambda\mu^{-1}c^{-2}a^2(t))^{-1/2}\times a^{-1}(t)dt\right]. \tag{16}$$

The minimum of (14) is obtained for $\eta=1/2R$,

$$\frac{r^2(t)}{R^2}=\frac{1-\sqrt{1-R^2/m^2}}{1+\sqrt{1-R^2/m^2}}. \tag{17}$$

For $m=\infty$ the shape of the trajectory in B^3 is a straight line through the center of B^3. In fact, using homogeneity and isotropy, the easiest way to calculate (14) is to do it for the case $m=\infty$, and then to transport the time-parametrized straight line by an element of the invariance group SL(2, $\mathbb{C}$) of (4) in any wished position.

For both $\eta\to\infty$ and $\eta\to 0$ we have $r(t)/R\to 1$. If $\lambda=0$ in (15) we get the characteristics of the eikonal equation [see Eq. (19)], the rays are geodesics independent of $\tilde{a}(t')$ in (3).

We discuss at first the massive case, $\lambda=1$ in (16). If $M=0$, $\mu=0$ in (13), η is constant, the particle is at rest, and we assume from now on that $\mu>0$.

The static case, $a(t)=1$, can be regarded as the special relativistic generalization of our example in Ref. 1; η varies between 0 and ∞, the trajectory starts at the boundary S_∞ of B^3 at $t=-\infty$ and again reaches S_∞ at $t=+\infty$.

In the case of de Sitter space $a(t)=\sinh(\Lambda t)$, $R=c/\Lambda$, $0\leqslant t\leqslant\infty$, we have for $t\to 0$, $\eta\sim\text{const.}(\Lambda t)^{\pm 1}$, i.e., η approaches either 0 or infinity; for $t\to\infty$, η approaches a finite constant. The same holds true for rays, $\lambda=0$ in (16).

For $a(t)=\Lambda t$, $0\leqslant t\leqslant\infty$, we have if $t\to 0$ again $\eta\sim\text{const.}(\Lambda t)^{\pm 1}$, and for $t\to\infty$ $\eta\sim\text{const.}\exp(\pm\mu^{1/2}c/t\Lambda)$, η approaches a finite constant. If $\lambda=0$ we have also for $t\to\infty$ $\eta\sim\text{const.}(\Lambda t)^{\pm 1}$.

For $t\to 0$ the curvature radius of three-space goes to zero, the Gaussian curvature gets infinite, and the distance between two interior points shrinks to zero. For $t\to\infty$ the distance between two points of B^3 gets infinite, the Gaussian curvature goes to zero.

We call a trajectory bounded for $t\to\infty$ if $r(t=\infty)/R<1$, corresponding to a finite η, $0<\eta(t=\infty)<\infty$. Likewise for $t\to 0$, if $r(t=0)/R<1$. If we do not specify in the following the direction of time, "bounded" refers to both directions of the time evolution. Note that in the last two examples (but not in the static case) the motion of massive particles is bounded for $t \to \infty$, but not uniformly, by a suitable choice of the integration constants we can make the ratio $r(t=\infty)/R$ arbitrarily close to 1.

In the case of light rays Eq. (13) is replaced by

$$m/R = \sqrt{1 + (\omega R / c\tilde{M})^2}, \tag{18}$$

where ω and $\tilde{M}$ are the integration constants in the eikonal $\psi(t,r,\varphi) = -\omega t + \tilde{M}\varphi + \tilde{\psi}(r)$. The eikonal equation reads in metric (3)

$$-\left(\frac{\partial \psi}{\partial t'}\right)^2 + \frac{1}{4}\left(1 - \frac{r^2}{R^2}\right)\left[\left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \psi}{\partial \varphi}\right)^2\right] = 0, \tag{19}$$

it does not contain $\tilde{a}(t')$. Its characteristics are given in (12), (14), and (15) with $\lambda = 0$.

Finally we discuss energy in these time-dependent metrics. Using polar coordinates and the line element (3), (4), we have for the space component $\mathbf{p}$ of the four-momentum

$$p_\mu = (p_0, \mathbf{p})$$
$$\mathbf{p}^2 = 4\tilde{a}^2(t)(1 - r^2/R^2)^{-2} m^2 c^2 \dot{t}^2(s)[r'^2(t) + r^2\varphi'^2(t)]. \tag{20}$$

Using (8) ($\lambda = 1$) and (11) we get[2,19,3]

$$\mathbf{p}^2 \tilde{a}^2(t') = m^2 c^4 \mu, \tag{21}$$

and from $p^\mu p_\mu = -m^2c^2$ we have

$$[c^{-2}E^2(t')\tilde{a}^2(t') - m^2c^2]\tilde{a}^2(t') = m^2c^4\mu. \tag{22}$$

In terms of the line element (1), (4) Eq. (21) reads

$$\mathbf{p}^2 a^2(t) = m^2 c^4 \mu, \tag{23}$$

and instead of (22) we have

$$E^2(t) = m^2c^6\mu a^{-2}(t) + m^2c^4. \tag{24}$$

Velocity we can define via $\mathbf{p} = m\mathbf{v}/\sqrt{1 - \mathbf{v}^2/c^2}$, for $a(t) \to \infty$, $\mathbf{v}$ approaches zero, for $a(t) \to 0$ the speed of light. Using Einstein's or de Broglie's relation we get from (23) the law for the variation of the wavelengths, $\lambda / a(t) = \text{const.}$,[7,8] that determines the red shifts.

Geodesic motion on the four-manifold $\mathbb{R}^{(+)} \times I \times S$ is realized by projections of geodesics of its universal cover $\mathbb{R}^{(+)} \times B^3$ (via the natural covering projection π: $\mathbb{R}^{(+)} \times B^3 \to \mathbb{R}^{(+)} \times I \times S$, cf. Refs. 12 and 13).

A geodesic in $\mathbb{R}^{(+)} \times B^3$ is an arc of a semicircle (12) with the time parametrization (14). The projection is realized by mapping every piece of this arc lying in a tile $\gamma(F)$ of the tesselation $\Gamma(F)$ (cf. Sec. II) via γ^{-1} into the polyhedron F. The face identification of F by Γ also gives the identification of the projected arc pieces, which inherit the time evolution of the covering geodesics.

If the initial ($t = 0$, $t = -\infty$ in the static case) or end point ($t = \infty$) of the arc representing the $\mathbb{R}^{(+)} \times B^3$ geodesic in B^3 lies in the limit set $\Lambda(\Gamma)$ (cf. Sec. II) of the tiling, it intersects infinitely many images $\gamma(F)$ of the tesselation. In this case the projected arc pieces in F are uniformly separated from S_∞, lying in a ball of finite hyperbolic diameter, cf. Ref. 1. If, on the other hand, the initial or end point of the arc lies in S_∞ but not in $\Lambda(\Gamma)$, it intersects only finitely many tiles, and the projected trajectory reaches S_∞ at $t = 0$ ($t = -\infty$) or $t = +\infty$ in the image of the last intersected tile.

Thus the condition that a trajectory in $\mathbb{R}^{(+)} \times I \times S$ is bounded is that the initial and end points of its covering trajectories (geodesics in $\mathbb{R}^{(+)} \times B^3$ whose covering projections give rise to the $\mathbb{R}^{(+)} \times I \times S$ trajectory) are either interior points of B^3 or boundary points lying in the limit set $\Lambda(\Gamma)$. This means that in the static case $a(t) = 1$ in (3) the bounded trajectories are just those whose covering trajectories have their initial ($t = -\infty$) and end point ($t = +\infty$) in $\Lambda(\Gamma)$. In the other two examples $a(t) = \sinh(\Lambda t)$ and $a(t) = \Lambda t$ boundedness requires only that the covering trajectories have their initial point at $t = 0$ in $\Lambda(\Gamma)$.

The condition that the initial or end point of the covering trajectories lies in $\Lambda(\Gamma)$ gives rise to strong chaotic behavior of their projections: they are recurrent, ergodic, have the Bernoulli property, and the Hausdorff dimension of $\Lambda(\Gamma)$ is linked with their topological entropy, cf. Ref. 20.

IV. WAVE MECHANICS: THE EMERGENCE OF LOCALIZED STATES

The wave equation in the covering space $\mathbb{R}^{(+)} \times B^3$ of the manifold $\mathbb{R}^{(+)} \times I \times S$ (cf. Sec. II) may be written as[22,23]

$$[\Box - (mc/\hbar)^2 - \xi\hat{R}\,]\psi = 0. \tag{25}$$

Here ξ is a dimensionless parameter that couples ψ to the curvature scalar $\hat{R}$ in (6). The wave operator[3] reads in terms of the line element (1), (4)

$$\Box = -c^{-2}a^{-3}(t)\frac{\partial}{\partial t}\left[a^3(t)\frac{\partial}{\partial t}\right] + \frac{1}{a^2(t)}\Delta_{B^3}, \tag{26}$$

with the Laplace–Beltrami operator of the Poincaré ball B^3

$$\Delta_{B^3} = \frac{1}{4}\left(1 - \frac{r^2}{R^2}\right)^2\left[\Delta_{E^3} + 2R^{-2}\left(1 - \frac{r^2}{R^2}\right)^{-1} r\frac{\partial}{\partial r}\right], \tag{27}$$

where Δ_{E^3} is the Euclidean Laplace operator.

We obtain wave mechanics on $\mathbb{R}^{(+)} \times I \times S$ by imposing periodic boundary conditions on the wave field in (25) with respect to the polyhedron F (cf. Sec. II), that represents three-space $I \times S$ in B^3. Here, ψ shall be periodic in B^3 under the discrete group Γ that is generated by the face-identifying transformations of F: $\psi(t,\gamma\mathbf{x}) = \psi(t,\mathbf{x})$ in $\mathbb{R}^{(+)} \times B^3$ for all elements γ of Γ.

Finally we will assume for ψ an end-value condition concerning the time dependence of ψ for $t \to +\infty$. The special choice of this condition depends on the factor $a(t)$ in (1), the time dependence of $\psi(t,\mathbf{x})$ for $t \to +\infty$ should approach as well as possible that of Minkowski space. Examples will be given in Sec. V.

Separation of variables in (25), $\psi(t,\mathbf{x}) = \varphi(t)\hat{\psi}(\mathbf{x})$, gives

$$[\Delta_{B^3} + \lambda\,]\hat{\psi} = 0, \tag{28}$$

and

$$\ddot{\varphi} + 3\dot{a}(t)a^{-1}(t)\dot{\varphi}(t) + [(mc^2/\hbar)^2 + c^2\lambda a^{-2}(t) + c^2\xi\widehat{R}(t)]\varphi = 0, \quad (29)$$

where λ is the separation parameter.

The spectral problem (28) with $\hat{\psi}$ periodic under Γ in B^3 has been discussed in the case of nonrelativistic motion[24,15,25,1]: we have absolutely continuous spectrum in the range $]\,1/R^2, \infty\,[$ and one square-integrable, localized solution for a λ_0 in $]0, 1/R^2\,]$. This λ_0 is connected with the Hausdorff dimension δ of the limit set $\Lambda(\Gamma)$ of the group Γ of covering transformations via the formula (cf. Refs. 25,26)

$$\lambda_0 = R^{-2}\delta(2-\delta). \quad (30)$$

To recapitulate what has already been stated in Sec. II: We start with a polyhedron F in B^3 and a face identification that gives rise to the topology of $I \times S$. The metric of B^3 is induced on F. We may deform the shape of F a little. This causes a slight perturbation of the face-identifying transformations, so that the group $\widetilde{\Gamma}$ generated by the perturbed generators is still isomorphic to Γ, but in general not SL(2, $\mathbb{C}$)-conjugated to Γ. The limit sets $\Lambda(\Gamma)$ and $\Lambda(\widetilde{\Gamma})$ have then different Hausdorff dimensions; the deformed polyhedron inherits a metric of B^3 that is globally nonisometric to that of F. Here, δ depends only on this metric, it may range in the interval $[1, 2[$.

The localized $\hat{\psi}$ corresponding to $\lambda_0 = R^{-2}\delta(2-\delta)$ admits a Helgason representation in B^3 as[25-27]

$$\hat{\psi}(\mathbf{x}) = \int_{\Lambda(\Gamma)} \frac{(1-|\mathbf{x}|^2/R^2)^\delta}{|\mathbf{x}-\boldsymbol{\eta}|^{2\delta}}\, d\mu(\boldsymbol{\eta}), \quad (31)$$

where $d\mu(\boldsymbol{\eta})$ is the Hausdorff measure on $\Lambda(\Gamma)$. (It is a conformal density of weight δ, with respect to Γ, that causes the periodicity of $\hat{\psi}$, cf. Ref. 20.) The wave field $\hat{\psi}$ is square integrable over F with the volume element of (4)

$$dV_{B^3} = 8(1-|\mathbf{x}|^2/R^2)^{-3}\, dx^3. \quad (32)$$

The spectral resolution of (28) corresponding to the continuous part of the spectrum in $]\,1/R^2, \infty\,[$ can be done in terms of Eisenstein series (cf. Refs. 28 and 29), but we will not make explicit use of them in the following.

Next we discuss the time evolution of ψ determined by Eq. (29), which is linear and second order. The Wronskian determinant of two solutions $\varphi_1\varphi_2$ depends up to a constant factor only on the coefficient in front of $\dot{\varphi}$, which means in the case of (29)

$$\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1 = \text{const.}\, a^{-3}(t). \quad (33)$$

We can define a covariant scalar product for solutions ψ_1, ψ_2 of (25) as

$$(\psi_1,\psi_2) = \frac{i}{2}\int_\Sigma \left(\psi_1 \frac{\partial}{\partial x_\mu}\overline{\psi_2} - \overline{\psi_2}\frac{\partial}{\partial x_\mu}\psi_1\right) d\Sigma^\mu, \quad (34)$$

where $d\Sigma^\mu$ is the volume element of a future directed spacelike hypersurface Σ in $\mathbb{R}^{(+)} \times I \times S$. The integral in (34) is independent of the special choice of Σ (via Green's theorem, cf. Refs. 23 and 3). If we use in (34) for Σ three-space, Σ^0 is the only nonvanishing component of $d\Sigma^\mu$ and is given by

$$d\Sigma^0 = a^3(t)\, dV_{B^3}. \quad (35)$$

Thus (34) is independent of time.

The normalization condition

$$(\psi,\psi) = \pm 1 \quad (36)$$

now reads for $\psi(t,\mathbf{x}) = \varphi(t)\hat{\psi}(\mathbf{x})$

$$\int_F \hat{\psi}\bar{\hat{\psi}}\, dV_{B^3} = 1 \quad (37)$$

and

$$\tfrac{1}{2}(\bar{\varphi}\dot{\varphi} - \dot{\varphi}\dot{\bar{\varphi}}) = \pm i a^{-3}(t). \quad (38)$$

The energy of a solution of (25) is determined via the T_{00} component of the energy-momentum tensor of the field. We consider the minimally coupled case $\xi = 0$ in (25). Then $T_{\mu\nu}$ is given by[23]

$$T_{\mu\nu} = \tfrac{1}{2}[\psi_{,\mu}\overline{\psi_{,\nu}} + \overline{\psi_{,\mu}}\psi_{,\nu}] + \tfrac{1}{2}Lg_{\mu\nu}, \quad (39)$$

with the Lagrange function

$$L = -\psi_{,\mu}\overline{\psi_{,\nu}}g^{\mu\nu} - (mc/\hbar)^2\psi\bar{\psi}, \quad (40)$$

which leads to (25):

$$\psi_{;\mu\nu}g^{\mu\nu} - (mc/\hbar)^2\psi = 0. \quad (41)$$

Applying (26), (41), and Green's formula to T_{00} in (39), we get

$$T_{00} = c^{-2}\hat{\psi}\bar{\hat{\psi}}\left\{\frac{1}{2}\frac{d\varphi}{dt}\frac{d\bar{\varphi}}{dt} - \frac{1}{4}\left[a^{-3}(t)\bar{\varphi}\frac{d}{dt}\left(a^3(t)\frac{d\varphi}{dt}\right) + \text{c.c.}\right]\right\} + O(\text{surface terms}). \quad (42)$$

Positivity of T_{00} follows from $\bar{\psi}\Box\psi - (mc/\hbar)^2\bar{\psi}\psi = 0$ and $\lambda > 0$ in (28).

Energy is now defined as the zero component of the four-momentum,

$$E = \hbar c^2 \int_\Sigma T_{0\mu}\, d\Sigma^\mu, \quad (43)$$

where we can take for Σ any spacelike hypersurface as in (34). If we choose for Σ three-space, cf. (35), and the normalization condition (36) (which we can also impose on wave fields of the continuous spectrum, by integrating in (37), (43) at first over a finite domain of F, and performing later the limits) the integration in (43) is trivial and we get

$$E(t) = \frac{1}{2}\hbar a^3(t)\left\{\frac{d\varphi}{dt}\frac{d\bar{\varphi}}{dt} - \frac{1}{2}\left[a^{-3}(t)\bar{\varphi}\frac{d}{dt}\left(a^3(t)\frac{d\varphi}{dt}\right) + \text{c.c.}\right]\right\}. \quad (44)$$

In this expression the space part $\hat{\psi}$ of the wave function does not enter, but via φ the spectral variable λ in (28), (29).

V. EXAMPLES

A. The static case $a(t) = 1$ in (1)

The time dependence of the positive/negative frequency solutions of (25) is determined by Eq. (29). Using the normalization condition (36), (38) we have

$$\varphi^+ = \omega^{-1/2}\exp(-i\omega t), \quad \varphi^- = \overline{\varphi^+}, \quad (45)$$

with

$$\omega = [(mc^2/\hbar)^2 + c^2\delta(2-\delta)/R^2]^{1/2}. \quad (46)$$

In (29) we have put $\xi = 0$ and λ as in (30). The localized wave ψ is in this static metric the ground state with the energy

$$E(\delta)=\hbar\omega(\delta)=mc^2\left[1+\frac{\hbar^2\delta(2-\delta)}{m^2c^2R^2}\right]^{1/2}, \qquad (47)$$

calculated from (44).

For the energy gap between $E(\delta)$ and the lower edge of the continuous spectrum located at $\delta=1$ (cf. Sec. IV) we have

$$E(\delta)-E(\delta=1) = -\frac{1}{2}\frac{\hbar^2}{mR^2}\left\{(\delta-1)^2+\frac{1}{4}\frac{\hbar^2}{m^2c^2R^2} \times\delta^2(2-\delta)^2+O\left(\frac{1}{c^4}\right)\right\}. \qquad (48)$$

The first term in (48) corresponds to the nonrelativistic case already discussed in Ref. 1. If $m=0$ we get instead of (47)

$$E(\delta)=(\hbar c/R)\sqrt{\delta(2-\delta)}. \qquad (49)$$

In the conformally coupled case, $\xi=1/6$, $m=0$, a ground state does not exist, because it is then impossible to generate wave motion in the late stages of the time evolution by any combination of the fundamental solutions of Eq. (29), compare the discussion of the following example.

B. De Sitter space: $a(t)=\sinh(\Lambda t)$, $0\leqslant t\leqslant\infty$

In this metric the curvature scalar in (6) is a constant, $\hat{R}=12\Lambda^2/c^2$, and we put in (4) $R=c/\Lambda$. Λ is connected with the cosmological constant, cf. Refs. 2, 19, 3, and 4. By introducing a new time variable in (29), compare (3),

$$t'=-\int^t a^{-1}(t)dt,\quad \sinh(\Lambda t)=\sinh^{-1}(\Lambda t'), \qquad (50)$$

and rescaling, $\varphi(t)=\sinh(\Lambda t')\tilde{\varphi}(t')$, we get for (29)

$$\frac{d^2}{dt'^2}\tilde{\varphi}(t')+\{\Lambda^2[\delta(2-\delta)-1]+\sinh^{-2}(\Lambda t') \times[(mc^2/\hbar)^2+12(\xi-1/6)\Lambda^2]\}\tilde{\varphi}=0. \qquad (51)$$

By trying the ansatz $\tilde{\varphi}(t')=\sinh^{1/2+\alpha}(\Lambda t')\times f(\sinh^2(\Lambda t'))$, $\sigma=\sinh^2(\Lambda t')$, we obtain for $f(\sigma)$ a hypergeometric differential equation if we choose for α

$$\alpha^2=\tfrac{1}{4}+12(\tfrac{1}{6}-\xi)-(mc^2/\Lambda\hbar)^2. \qquad (52)$$

In the following we assume $\alpha^2<0$, $\alpha=:i\kappa$. Furthermore we define

$$\nu^2:=1-\delta(2-\delta). \qquad (53)$$

A fundamental system of solutions of (29) is then given by

$$\varphi(t)=(\Lambda|\kappa|)^{-1/2}\sinh^{-3/2}(\Lambda t)\cosh^{-i\kappa}(\Lambda t) \times\tanh^{\nu+1/2}(\Lambda t)H(\Lambda t),$$
$$H(\Lambda t):={}_2F_1(\tfrac{1}{4}+(\nu+i\kappa)/2,\ \tfrac{3}{4}+(\nu+i\kappa)/2,\ 1+i\kappa;\ \cosh^{-2}(\Lambda t)), \qquad (54)$$

and its complex conjugate $\overline{\varphi(t)}$. The solution in (54) we normalized according to (38). $\varphi(t)$ is invariant with respect to a change of sign of ν (which is either real if $1\leqslant\delta<2$, or imaginary, if $\delta=1+is$, $0\leqslant s\leqslant\infty$, parametrizing the continuous spectrum). A change of sign of κ corresponds to complex conjugation.

The solution (54) admits the asymptotic expansion (for $t\to\infty$)

$$\varphi(t)=(\Lambda|\kappa|)^{-1/2}2^{(3/2+i\kappa)}e^{-\Lambda t(3/2+i\kappa)} \times\left\{1+\frac{e^{-2\Lambda t}}{1+i\kappa}\left[\frac{5}{4}+\nu^2+\frac{3}{2}i\kappa\right]+O(e^{-4\Lambda t})\right\}. \qquad (55)$$

For $t\to0$ and $\mathrm{Re}(\nu)>0$ we have $\varphi(t)\sim c_0t^{-\nu-1}$; if $\mathrm{Im}(\nu)>0$ we obtain $\varphi(t)\sim t^{-1}[c_1\cos(|\nu|\log\Lambda t)+c_2\sin(|\nu|\log\Lambda t)]$, with constants c_0, c_1, c_2 depending on ν and κ.

Inserting (55) into (44) we have for $t\to\infty$ (and $\xi=0$ in κ)

$$E(t,\nu)=(\hbar\Lambda/|\kappa|)(\kappa^2+\tfrac{9}{4})+O(e^{-2\Lambda t}) =mc^2+O(\hbar^2,e^{-2\Lambda t}), \qquad (56)$$

and

$$E(t,\nu)-E(t,\nu=0) =[\hbar\Lambda\nu^2e^{-2\Lambda t}/|\kappa|(1+\kappa^2)]\,[(2\kappa^2-\tfrac{11}{2})\cos(\kappa\Lambda t) -\kappa(\tfrac{19}{2}+2\kappa^2)\sin(\kappa\Lambda t)]+O(e^{-4\Lambda t}). \qquad (57)$$

For $t\to0$ and $\mathrm{Re}(\nu)>0$ we have $E(t,\nu)\sim c_3t^{-2\nu-1}$, e.g., $E(t,\nu=\tfrac{1}{2})\sim\tfrac{3}{2}(\hbar\Lambda/|\kappa|)(\Lambda t)^{-2}$, and for $\mathrm{Im}(\nu)>0$, the continuous spectrum, we get

$$E(t,\nu)\sim t^{-1}[c_4+c_5\cos(|\nu|\log\Lambda t)+c_6\sin(|\nu|\log\Lambda t)],$$

with constants depending on ν and κ, so that $E(t,\nu)>0$.

Discussion: The boundary condition concerning the time dependence of the solution of (25) is that the negative/positive frequency solutions shall behave like $f(t)\exp(\pm ig(t))$ for $t\to\infty$ [with $f(t)\to0$ and $g(t)\to+\infty$ if $a(t)\to\infty$ in (1)], when three-space gets flat and massive particles come to rest. This is only possible if $\alpha^2<0$ in (52). Thus in the minimally coupled case $\xi=0$ m must exceed a threshold value, $mc^2/\Lambda\hbar>\tfrac{3}{2}$, and if $m=0$, ξ must exceed $\tfrac{3}{16}$, excluding the case of conformal coupling ($\xi=\tfrac{1}{6}$).

In the static metric the spectral variable λ in (28) is directly related to energy by Eq. (47). This is not any more so in the time-dependent case. Here λ, which enters via ν in (53), plays the same part as the dimensionless ξ in (25), parametrizing the possible solutions of (25) under the given boundary conditions. The only restrictions for ξ and λ are $\alpha^2(\xi,m)<0$, and that λ ranges over the spectrum of the space part of the d'Alembertian in (26).

If we compare the energy of the wave field $E(t,\nu)$ with the classical formula (24), we have for $t\to\infty$ essentially the same time dependence, with a slightly modified rest energy in the quantum case. For $t\to0$ we also have

$$E_{\text{quantum}}/E_{\text{classical}}=O(1).$$

The parameters λ and ξ take over the part of μ in (24), determined by the classical initial conditions via (13).

Whatever the choice of λ, for $t\to\infty$ when the energies approach their rest value, their difference gets of the order $O(a^{-2}(t))$.

Nevertheless there is a qualitative difference in the time behavior of the energy of a localized state and that of an unbounded state in the early phase of the expansion. In the case of a square-integrable wave function, cf. (32), the main part of the density $|\psi|^2$ stays away from the boundary of three-space (the open ends of F on S_∞, cf. Sec. II), and for $a(t)\to 0$ $|\psi|^2$ gets concentrated, for the distances between interior points of the spacelike slices $t=$ const. go to zero. The price to pay for that is localization energy. On the other hand, if $a(t)\to\infty$ the distances between interior points increase to infinity, ψ gets delocalized, resembling more and more an unbounded state.

C. A universe that is Minkowskian with the topology of $\mathbb{R}^+\times I\times S$

If we take in (1) the linear expansion factor $a(t)=\Lambda t$, $R=c/\Lambda$ in (4), $0\leqslant t\leqslant\infty$, the curvature scalar in (6) vanishes. Indeed, changing coordinates in (1), (4), cf. Ref. 2,

$$r'=2ct\frac{r/R}{1-r^2/R^2},\quad t'=t\frac{1+r^2/R^2}{1-r^2/R^2},\tag{58}$$

we get the Minkowski line element in polar coordinates. But in these coordinates space and time get mixed up in the boundary conditions, and the wave equation is not separable.

Defining ν as in (53), we get as a normalized fundamental system of Eq. (29) with $\lambda=\Lambda^2c^{-2}\delta(2-\delta)$ as in the previous example

$$\varphi(t)=(\pi/2)^{1/2}e^{-i\pi\nu/2}\Lambda^{-3/2}t^{-1}H_\nu^{(2)}(mc^2t/\hbar),\tag{59}$$

and its complex conjugate. ($H_\nu^{(2)}$ is a Hankel function,[30] the phase factor is needed if $\nu^2<0$.)

The asymptotic expansion of (59) to the order we need is

$$\varphi(t)=e^{i\pi/4}(\hbar/mc^2)^{1/2}(\Lambda t)^{-3/2}e^{-imc^2t/\hbar}\times[1+iB(\nu)/t+C(\nu)/t^{-2}+O(t^{-3})],\tag{60}$$

with

$$B(\nu)=-\frac{1}{2}\frac{\hbar}{mc^2}\left(\nu^2-\frac{1}{4}\right),$$

$$C(\nu)=-\frac{1}{8}\left(\frac{\hbar}{mc^2}\right)^2\left(\nu^2-\frac{1}{4}\right)\left(\nu^2-\frac{9}{4}\right).$$

For $t\to\infty$ we have with (44), (60)

$$E(t,\nu)=mc^2+O(t^{-2}),\tag{61}$$

and

$$E(t,\nu)-E(t,\nu=0)=-\tfrac{1}{4}(\hbar^2/mc^2)t^{-2}\nu^2(\nu^2+15)+O(t^{-4}).\tag{62}$$

For $t\to 0$ we have the same qualitative behavior of φ and E as in de Sitter space $[a(t)=\Lambda t\approx\sinh(\Lambda t)]$. If $\nu>0$,

$$E(t,\nu)\sim(1/2\pi)(1+\nu)2^{2\nu}\Gamma^2(\nu)mc^2(mc^2t/\hbar)^{-2\nu-1},$$

e.g., $E(t,\nu=\frac{1}{2})\sim 3/2mc^2(mc^2t/\hbar)^{-2}$, and for the continuous spectrum $\mathrm{Im}(\nu)>0$

$$E(t,\nu)\sim t^{-1}[c_1+c_2\cos(|\nu|\log mc^2t/\hbar)+c_3\sin(|\nu|\log mc^2t/\hbar)].$$

All that has been stated about the de Sitter example applies here too, with obvious modifications. So the parameter ξ in (25) does not enter here at all, because $\widehat{R}=0$, and the exponential decay in (57) is replaced by the algebraic one in (62), as it is in $a^{-2}(t)$.

In the case of $m=0$ we have for the positive frequency solution of (29)

$$\varphi(t)=\Lambda^{-1/2}\kappa^{-1/2}(\Lambda t)^{-1}e^{-i\kappa\log\Lambda t},\tag{63}$$

with $\kappa=[\delta(2-\delta)-1]^{1/2}>0$. For $E(t)$ we have from (63)

$$E(t)=\hbar t^{-1}(\kappa+\kappa^{-1})\tag{64}$$

and $E(t)a(t)=$ const., compare the red shift relation at the end of Sec. III. Since κ must be positive that φ has the correct time behavior for $t\to\infty$, δ cannot lie in the interval $[1,2[$, which excludes a localized wave solution in the case of zero rest mass in this universe, compare our discussion of de Sitter space. The energy asymptotics of models whose expansion factors have power law behavior in the asympototic regions is discussed in Refs. 9 and 10.

VI. CONCLUSION

Open RW cosmologies of multiple spatial connectivity show localization phenomena foreign to the simply connected open standard models of cosmology.

In the case of classical geodesic motion they manifest themselves in the appearance of bounded chaotic trajectories, in quantum mechanics as localized wave fields. These phenomena are purely topological; the fact that a trajectory is bounded or not does not depend on its energy (there is no threshold value as in Hamiltonian mechanics), but is decided in the universal covering space of the manifold. If its covering trajectories emerge from the limit set of the covering group it is bounded and chaotic, otherwise not.

If the metric is time dependent we no longer have a simple relation between the spectrum of the wave equation and the energy of the corresponding quantum states. Nevertheless there is a qualitative difference in the time evolution of the energy of a localized state and that of an unbounded one, during the early stages of the expansion of the universe (see the discussion of de Sitter space), but also there does not exist something like a threshold value.

On the other hand, the Hausdorff measure of the limit set $\Lambda(\Gamma)$ on the boundary of the universal covering space determines the spatial part of the localized wave field. The Hausdorff dimension of $\Lambda(\Gamma)$ determines the time asymptotics of the energy of the field at the beginning of the expansion. Thus this fractal, quasi-self-similar set provides without any (semi-classical) approximation the link between the collection of the bounded chaotic trajectories and the localized wave field on the manifold.

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