RELATIVISTIC CHAOS IN ROBERTSON-WALKER COSMOLOGIES: THE TOPOLOGICAL STRUCTURE OF SPACE-TIME AND THE MICROSCOPIC DYNAMICS

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I. INTRODUCTION

Among the general relativistic models that attempt to describe the rough overall structure of the universe and its origins , Robertson-Walker geometries are today regarded as the most likely candidates. Appealing to the principles of homogeneity and isotropy, a R.-W. cosmology is locally described by a line element $ds^2 = -dt^2 + a^2(t)\, d\sigma^2$, where $d\sigma^2$ is the line element of of a 3-space of constant curvature. The cosmological expansion factor a(t) determines the time dependence of the frequencies of light in the universe, and is according to the observed red-shifts an increasing function of time. The second fundamental property of a(t) is that it determines via Einstein's equations together with the Gaussian curvature of 3-space the energy density ε and the pressure p of the light-matter content of the universe.

Though Einstein's equations give the relation between ε, p and a(t), they do not determine the global topological structure of the universe, in particular they say nothing about the topology of the spacelike 3-sections at a fixed instant of time. Homogeneity and isotropy demand that these sections are 3-manifolds of constant curvature, but nothing prevents them from having different geometries and even topologies at two different instants of time.

Traditionally three different geometries are considered: in the case of zero curvature the 3-sections are Euclidean 3-space , for positive curvature one considers the 3-sphere, for negative curvature a shell of the Minkowski hyperboloid. The topology of 4-space is then the product of the real line or a semi-infinite interval with one of these topologies. Moreover different spacelike sections at different instants of time are isometric after a simple rescaling. The spacelike slices in the case of positive curvature are compact and such cosmologies are called closed, the other two cases correspondingly open.

These three examples do by far not exhaust all possible topologies of 3-manifolds of constant curvature, and in [14] we started to investigate the influence of a possible non-trivial topology of the spacelike slices on the microscopic dynamics. Which topologies come in question? Three-dimensional manifolds of constant zero or positive curvature are rather exceptional, cf.[9], similar to positively curved or flat Riemann surfaces. The generic case are manifolds of negative curvature, called hyperbolic. Such manifolds are best imagined in hyperbolic space as polyhedra with their faces identified in pairs, cf.[5], analogous to the 2-torus as a square in the Euclidean plane. As a representation of hyperbolic space we may choose the Poincaré ball B^3, $|\vec{x}| < R$, $d\sigma^2 = (1 - |\vec{x}|^2/R^2)^{-2} d\vec{x}^2$. The geodesics in this

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ball are arcs of circles, and the the totally geodesic planes are spherical caps, both orthogonal to the sphere S_∞, the boundary of B^3. The polyhedral faces lie on geodesic planes and are identified in pairs by glueing mappings, elements of the group of isometries of B^3, which happens to be the Lorentz group, cf.[1]. The polyhedra may also have faces lying on S_∞ which are not identified in pairs, and which constitute the boundaries of the manifold. Such polyhedra have then infinite volume if measured by the Poincaré metric $d\sigma^2$ in B^3.

The glueing transformations generate a discrete subgroup Γ of the Lorentz group, and the images of the polyhedron F, $\Gamma(F)$ will create a tessellation of the Poincaré ball. There are accumulation points of this tiling on the boundary of B^3 (the fractal curves in Figs. 1,2) This limit set plays a large role in the spectral theory of the Laplace-Beltrami operator on the spacelike sections, for example its Hausdorff dimension determines the ground state eigenvalue, cf.[3, 8].

There are the compact hyperbolic manifolds, whose polyhedra do not touch the boundary of B^3, providing closed models of negative curvature. The five Platonic solids, regular polyhedra in B^3, are typical examples of them, cf.[5]. The most important feature of these finite-volume manifolds is that they are rigid [11], a given topology, defined by the face-pairing can carry only one metric of constant negative curvature. Therefore it is not possible to deform a Platonic solid a little, keeping a chosen face-identification, so that the deformed polyhedron with the deformed glueing mappings generate again a tiling of B^3. Thus finite-volume hyperbolic manifolds are also rather exceptional, being too rigid to be proper candidates for the spacelike slices.

The generic case of constant curvature 3-manifolds are thus hyperbolic manifolds of infinite volume, their spacelike slices being open, infinite. Here the topological structure does not fix the metric at all. The polyhedra, having free faces on S_∞, are together with the covering group Γ deformable without destroying their tiling property. Such deformations can be parametrized by a certain number (depending on the topology) of variables which characterize geometrically the polyhedron, cf.[2, 4, 13], and one obtains so an explicit realization of the deformation space of non-equivalent metrics on the topological manifold.The metric of B^3 is of course always induced on F. Thus a R.-W. cosmology is determined by the choice of an expansion factor a(t) in the line element ds^2, and by a path $(\Gamma(t), F(t))$, generically time-dependent, in the deformation space of an open hyperbolic 3-manifold, cf.[13, 15].

In [14] we started to analyse R.-W. cosmologies whose spacelike slices have a non-trivial topological structure, and the consequences that arise from the topology, both for classical world lines and for scalar quantum fields. In [16] we discussed de Sitter space, i.e. an expansion factor of the form $a(t) = \sinh(\Lambda t)$, and spacelike slices of the form $I \times S$, I an open finite interval, S a Riemann surface ('thickened surfaces'). We calculated the time evolution of the energy of the corresponding wave fields, and the bearing of the spectrum of the L.-B. operator of the 3-slices on this evolution.

In this paper we will give further examples, in particular we will figure out expansion factors a(t) that are compatible with the conditions of positive pressure and energy, $\varepsilon, p > 0$, (in de Sitter space we have $\varepsilon=p=0$), and we will discuss the energy asymptotics of the scalar wave fields in the asymptotically flat regime, for $t \to \infty$, and towards the initial singularity, for $t \to 0$.

II. THE ENERGY OF SCALAR WAVE FIELDS AND THE SPECTRUM OF THE LAPLACE - BELTRAMI OPERATOR OF THE SPACE SECTIONS

As pointed out in Section 1 the negative curvature (but not the finite or infinite volume

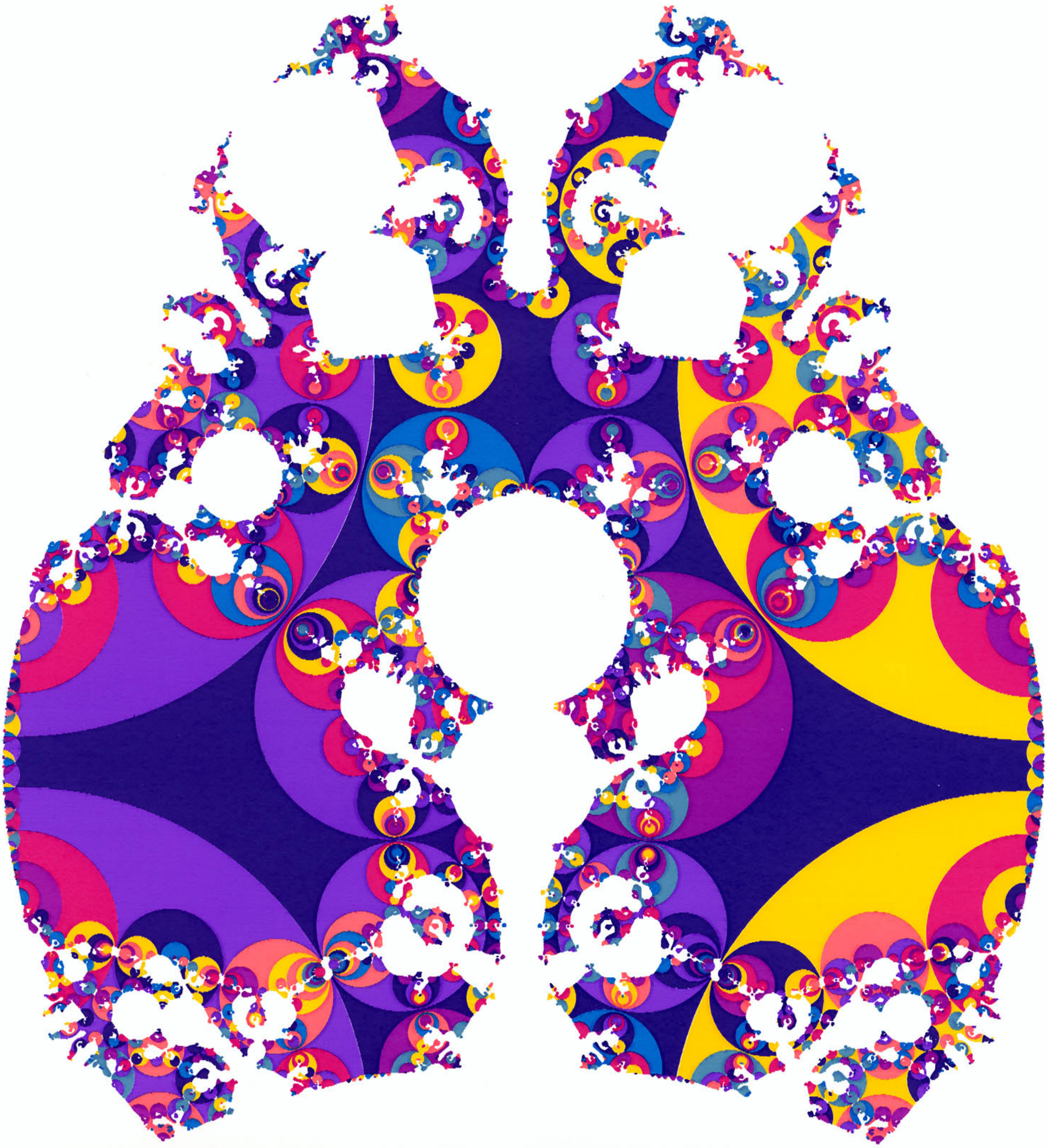

Fig.1

Fig.2

Fig.1. Tiling induced on the boundary S_∞ of the Poincaré ball B^3 by the universal cover of the 3-manifold $I \times S$. The convex hull of the fractal Jordan curve $\Lambda(\Gamma)$ determines a compact region $C(\Lambda)\backslash\Gamma$ (see Sec.3) in infinite 3-space F, where the chaotic trajectories lie. $g(S) = 19$, $\delta = 1.402 \pm 0.001$ (δ has been calculated by the method of characteristic curves, cf.[15]).

Fig.2. As Fig.1, covering of S_∞ stemming from a spacelike section F of the 4-manifold. Fig.1(b) in [14] - Fig.1 - Fig.2 - Fig.2 in [16] represent a sequence of non-isometric points on a path $(F(t), \Gamma(t))$ in the deformation space of the topological manifold $I \times S$. $g(S) = 19$, $\delta = 1.423$.

of the space sections) restricts crucially the possibilities of the asymptotic behaviour of the cosmological expansion factor a(t).

The energy density ε and the pressure p of the light-mass content of the universe are simple functions of a(t) and its derivatives, cf.[7],

$$\varepsilon = 3\frac{c^2}{8\pi k}\left[\frac{-\Lambda^2}{a^2}+\frac{\dot{a}^2}{a^2}-\frac{\hat{\lambda}c^2}{3}\right],\quad p=\frac{c^2}{8\pi k}\left[-2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}+\frac{\Lambda^2}{a^2}+\hat{\lambda}c^2\right], \tag{2.1}$$

and they have to be positive. In the following we put $c^2/8\pi k$ equal 1, $\hat{\lambda}$ is the cosmological constant.

We assume in this paper that a(t) is strictly increasing. Then, in the case of a positive $\hat{\lambda}$ the conditions $\varepsilon \geq 0$, $p \geq 0$ require exponential increase of a(t) for $t \to \infty$ as it happens in de Sitter space. If $\hat{\lambda}$ is negative a(t) cannot diverge for $t \to \infty$, in fact in this case we have an oscillating universe $a(t) = \sin(\Lambda t)$ which may be treated analogously to the examples given later on. From now on we assume $\hat{\lambda} = 0$. Then the positivity conditions require an asymptotically linear expansion factor $a(t) \sim \Lambda t$ for $t \to \infty$. Λ is a constant setting the time scale, a(t) is dimensionless. The case $a(t) = \Lambda t$ exactly has been sketched in [14], it leads to $\varepsilon=p=0$, but correction terms to this linear behaviour may change this situation considerably. Luckily the conditions $\varepsilon > 0$, $p > 0$ restrict the form of these correction terms too.

A. The asymptotically flat regime; time asymptotics for $t \to \infty$

We discuss a form of the expansion factor that exhausts qualitatively all possibilities of the asymptotic behaviour of the solutions of the wave equation for $t \to \infty$.

Example 1

$$a(t) = \Lambda t + c(\log\Lambda t)^{\alpha}. \tag{2.2}$$

From (2.1) we have

$$\varepsilon \sim 6\alpha c\Lambda^{-1}t^{-3}\left(\log\Lambda t\right)^{\alpha-1},\ p \sim 2c\alpha(1-\alpha)\Lambda^{-1}t^{-3}\left(\log\Lambda t\right)^{\alpha-2},\ \text{if } \alpha=1:\ p \sim 2c^2\Lambda^{-2}t^{-4}\log\Lambda t. \tag{2.3}$$

Thus to ensure $\varepsilon > 0$, $p > 0$ we have to require $0 < \alpha \leq 1$, $c > 0$, or $\alpha < 0$, $c < 0$. The case $\alpha = 1$, $c = 1$ appears in [6, p. 222].

With the a(t) in (2.2) we will now calculate solutions of the wave equation and the asymptotic behaviour of the energy evolution of these wave fields. The way to do this can be found in [14] in some detail, and we sketch it here very shortly to keep track with self-containedness. If we make a separation ansatz $\psi(t, \vec{x}) = \tilde{\psi}(\vec{x})\,\varphi(t)$ in the Klein-Gordon equation

$$\left[\Box - \xi R - (mc/\hbar)^2\right]\psi = 0, \tag{2.4}$$

we obtain for the time dependence of the wave fields

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \left[m^2 + \frac{\Lambda^2\lambda}{a^2} + c^2\xi R\right]\varphi = 0 \quad . \tag{2.5}$$

We write in (2.5) m for $mc^2/\hbar$; $\Box$ denotes the wave operator on the 4-manifold and R is its curvature scalar,

$$R = \frac{6}{c^2}\left[\frac{-\Lambda^2}{a^2} + \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right]. \tag{2.6}$$

ξ is a dimensionless coupling constant of the field to the scalar curvature, and Λ appears in the expansion factor. The radius of the Poincaré ball in Sec.1 we fix as c/Λ. Finally λ is the spectral parameter for the space part of the wave equation, and thus it varies over the spectrum of the Laplace-Beltrami operator of the spacelike slices of the 4-manifold. In the example of the manifold in Figs.1, 2 λ can admit a discrete value $\delta(2-\delta)$, δ the Hausdorff dimension of the fractal limit set $\Lambda(\Gamma)$, $1 \le \delta < 2$, which corresponds to the ground state eigenvalue of the L.-B. operator, and it can vary in $[1, \infty)$, corresponding to the continuous spectrum. We also assume that the 3-manifold (F, Γ) does not vary in time.

For the asymptotic expansions that we will carry out it is useful to eliminate the first derivative in (2.5) by introducing a new dependent variable $\psi := a^{3/2}\varphi$,

$$\ddot{\psi} + \left[m^2 + \frac{\Lambda^2}{a^2}(\lambda-6\xi) + \frac{\ddot{a}}{a}(6\xi-3/2) + \frac{\dot{a}^2}{a^2}(6\xi-3/4)\right]\psi = 0 \quad . \tag{2.7}$$

To obtain the right asymptotic behaviour of the energy , $E \sim mc^2$ in the limit $t \to \infty$, we have to impose the normalization condition

$$\dot{\psi}\bar{\psi} - \dot{\bar{\psi}}\psi = \pm 2i \tag{2.8}$$

on the solutions of (2.7).

Inserting a(t) of (2.2) into (2.5, 2.7) we calculate in the asymptotic order we need

$$\varphi = Am^{-1/2}(\Lambda t)^{-3/2} e^{imt}\left[1 - iBt^{-1} + Ct^{-2} + i\varepsilon Dt^{-2}(\log\Lambda t)^{\alpha} + O(t^{-2}(\log\Lambda t)^{\alpha-1})\right] \tag{2.9}$$

with

$$A = 1 - \frac{3}{2}\frac{\varepsilon}{\Lambda t}(\log\Lambda t)^{\alpha} + \frac{15}{8}\frac{\varepsilon^2}{(\Lambda t)^2}(\log\Lambda t)^{2\alpha} + O((\log\Lambda t)^{3\alpha}/(\Lambda t)^3) \quad ,$$

and $B = (\lambda-3/4)/2m$, $C = -(\lambda^2 + \lambda/2 - 15/16)/8m^2$. D is a real constant that does not enter in the final result for the energy.

In [16] the following formula for the energy of a wave field has been derived:

$$E = \frac{1}{2}\hbar a^3\left\{\dot{\varphi}\dot{\bar{\varphi}} + \varphi\bar{\varphi}\left[m^2 + (\lambda-6\xi)\Lambda^2 a^{-2} + 6\xi\frac{\dot{a}^2}{a^2}\right] + 6\xi\frac{\dot{a}}{a}(\dot{\varphi}\bar{\varphi} + \dot{\bar{\varphi}}\varphi)\right\}, \tag{2.10}$$

it is always positive definite for $0 \le 6\xi \le \min[1, \delta(2-\delta)]$, which can be seen easily by completing the first and the last term to a square. Inserting (2.9) into(2.10) we arrive at ($m \to mc^2/\hbar$)

$$E(\lambda) = mc^2A^2 + \frac{1}{2}\frac{\hbar^2}{mc^2}\frac{1}{t^2}(\lambda + \frac{9}{4} - 18\xi) + O((\log\Lambda t)^{\alpha}/t^3) \quad . \tag{2.11}$$

From (2.11) we have in 'universal' quantities

$$E(\lambda_1) - E(\lambda_2) \sim \frac{1}{2}\frac{1}{a^2(t)}\frac{\Lambda^2\hbar^2}{mc^2}(\lambda_1 - \lambda_2) \quad . \tag{2.12}$$

If m=0 the solution (2.9) has to be replaced by

$$\varphi = \Lambda^{-1/2}(\lambda - 1)^{-1/4}(\Lambda t)^{-1+i\sqrt{\lambda-1}}\left[1 + O((\log\Lambda t)^{\alpha}/t)\right], \tag{2.13}$$

with the energy

$$E = \frac{\hbar}{t}\left[\sqrt{\lambda-1} + (1-6\xi)/\sqrt{\lambda-1}\right] + O((\log\Lambda t)^{\alpha}/t^2) \quad . \tag{2.14}$$

The frequency of the oscillation of (2.13) is

$$\nu = (e^{2\pi/\sqrt{\lambda-1}} - 1)^{-1} t^{-1} , \tag{2.15}$$

and thus we may write (2.14) as

$$E \sim h(\lambda)\nu , \tag{2.16}$$

with $h(\lambda \to \infty) \to h$, $h(\lambda \to 1) \to \infty$. The asymptotic expansions are carried out for a fixed λ. In the massive case (2.9) we have of course $\nu = mc^2/h$.

An expansion factor of the form

$$a(t) = \Lambda t + c(\Lambda t)^{-\beta} \tag{2.17}$$

can be treated completely analogously to (2.2): we have for (2.17)

$$\varepsilon \sim -6c\beta\Lambda^2(\Lambda t)^{-\beta-3} , \; p \sim -2c\beta^2\Lambda^2(\Lambda t)^{-\beta-3} , \tag{2.18}$$

positivity is insured for $\beta > 0$, $c < 0$. One has to replace in the formulae of Ex.1 $(\log\Lambda t)^{\alpha}$ by $(\Lambda t)^{-\beta}$. In particular the formulae for the energy and the frequency remain in the order given exactly the same. Mixtures of powers and logarithms in a(t) do not alter anything qualitatively.

B. The approach to the initial singularity: time asymptotics of the fields and their energies for $t \to 0$

In the limit $t \to 0$ the expansion factor may go either to zero or approach a finite value. Exponential decay of a(t) violates $\varepsilon > 0$, $p > 0$, power law decay $a(t) \sim (\Lambda t)^{\alpha}$, $\alpha > 0$ leads to a positive energy density and pressure in the range $0 < \alpha \leq 2/3$,

$$\varepsilon \sim 3\alpha^2 t^{-2}, \; p \sim \alpha(2-3\alpha)t^{-2} \quad . \tag{2.19}$$

There is also the possibility $a(t) = \Lambda t$ strictly , without any corrections for $t \to 0$, that has been discussed in [14], and the two mentioned de Sitter examples. All these cases give $\varepsilon = p = 0$.

Logarithmic decay, $a(t) = (\log 1/\Lambda t)^{-\beta}$, $\beta > 0$, leads to

$$\varepsilon \sim 3\beta^2 t^{-2}(\log 1/\Lambda t)^{-2}, \; p \sim 2\beta t^{-2}(\log 1/\Lambda t)^{-1}. \qquad (2.20)$$

Models with a two-sided infinite time scale, $a(t \to -\infty)$ strictly decreasing, are incompatible with negative curvature, violating $\varepsilon > 0, p > 0$.

Example 2

$$a(t) = (\Lambda t)^{\alpha}, \; 0 < \alpha \leq 3/2, \; t \to 0; \qquad (2.21)$$

the following discussion holds even true for $0 < \alpha < 1$, we will comment on this in Ex.7.

The wave equation reads according to (2.7)

$$\ddot{\psi} + \left[m^2 + \hat{A}\, t^{-2} + \hat{B}\, \Lambda^2 (\Lambda t)^{-2\alpha}\right] \psi = 0, \qquad (2.22)$$

with $\hat{A} = 3\alpha(1 - 3\alpha/2)/2 + 6\xi\alpha(2\alpha - 1)$, $\hat{B} = \lambda - 6\xi$.

In the lowest order asymptotic expansion that we will use we can drop the m^2-term in (2.22), a fundamental system is then $t^{1/2 \pm \nu}$, $\nu := \sqrt{1/4 - \hat{A}} \geq 0$; $\nu^2 \geq 0$ is always satisfied for $0 < \alpha < 1$, $0 \leq 6\xi \leq 1$.

The normalized (see (2.8)) general solution of (2.5) is

$$\varphi = \left[\Lambda^{-1/2} A (\Lambda t)^{1/2 - 3\alpha/2 - \nu} + \Lambda^{-1/2} B (\Lambda t)^{1/2 - 3\alpha/2 + \nu}\right]\left[1 + O(t^{2(1-\alpha)})\right] \qquad (2.23)$$

with

$$A = a e^{i\varphi/2}, \; B = b e^{-i\varphi/2}, \; \sin\varphi = \pm 1/2\nu ab. \qquad (2.24)$$

In choosing A, B as we did we have fixed a constant overall phase factor which always drops out in terms like $\bar{\dot{\varphi}}\varphi$, $\varphi\bar{\dot{\varphi}}$, and so it does not affect the energy.

The solution (2.9) of Eq. (2.5) for $t \to \infty$ we fixed by imposing the end-value condition that for $t \to \infty$ φ should approach as closely as possible the Minkowski space solution. This fixes in principle also a and b in (2.24), which are functionals of the expansion factor a(t) via (2.5). The problem is of course that we do not know the function a(t) in the intermediate regime, only its asymptotic limits. Therefore we will discuss the time asymptotics of the energy leaving the numerical values of a, b undetermined. In the de Sitter example in Ref.[16] we knew the expansion factor in the intermediate region, and we could determine a, b by solving Eq.(2.5).

From (2.23) and (2.10) we get

$$E \sim \frac{1}{2}\hbar\Lambda |A|^2 (\Lambda t)^{-1-2\nu}\left[\left(-3\alpha/2 + 1/2 - \nu + 6\xi\alpha\right)^2 + \alpha^2 6\xi(1 - 6\xi)\right]. \qquad (2.25)$$

We assume as always $0 \leq 6\xi \leq \min[1, \delta(2-\delta)]$ to have the functional (2.10) positive definite. For $\xi = 1/6$ this expression vanishes, and we have to calculate higher orders. For $\xi = 0$ and $0 < \alpha \leq 1/3$ it vanishes likewise.

(a) $\xi = 1/6$, i. e. $\nu = (1-\alpha)/2$

Instead of (2.23) we have

$$\varphi = \Lambda^{-1/2}A(\Lambda t)^{-\alpha}\left[1 + c(\Lambda t)^{2(1-\alpha)} + O((\Lambda t)^{4(1-\alpha)})\right] + \Lambda^{-1/2}B(\Lambda t)^{1-2\alpha}\left[1 + O((\Lambda t)^{2(1-\alpha)})\right]; \tag{2.26}$$

c is a real constant, its numerical value does not enter in E which reads now

$$E \sim \frac{1}{2}\hbar\Lambda(\Lambda t)^{-\alpha}\left[(1-\alpha)^2|B|^2 + (\lambda-1)|A|^2\right]. \tag{2.27}$$

For small α the energy in (2.25) is a factor t^{-2} stronger divergent, compare also Eqs. (2.34, 2.36).

(b) $\xi = 0$, $0 < \alpha < 1/3$, $\nu = 1/2 - 3\alpha/2$

For the wave field and the energy we get in this case

$$\varphi = \Lambda^{-1/2}\left[A + B(\Lambda t)^{1-3\alpha}\right]\left[1 + O((\Lambda t)^{2(1-\alpha)})\right], \tag{2.28}$$

$$E \sim \frac{1}{2}\hbar\Lambda(\Lambda t)^{-3\alpha}|B|^2(1-3\alpha)^2, \tag{2.29}$$

the power in (2.25) would be $t^{-2+3\alpha}$, again a factor of t^{-2} stronger for small α.

(c) $\xi = 0$, $\alpha = 1/3$, $\nu = 0$

$$\varphi = \Lambda^{-1/2}\left[A + B\log\Lambda t\right]\left[1 + O((\Lambda t)^{4/3})\right], \tag{2.30}$$

$$E \sim \frac{1}{2}\hbar t^{-1}|B|^2, \tag{2.31}$$

A and B are connected here via $\sin\varphi = \pm 1/ab$, see (2.24).

Finally, from (2.25) it follows that the energy for $\xi = 0$, $1/3 < \alpha < 1$ is given by (2.29) with B replaced by A.

Example 3

$$a(t) = (\log 1/\Lambda t)^{-\beta},\ \beta > 0,\ t \to 0. \tag{2.32}$$

According to (2.20) we have $\varepsilon > 0, p > 0$. The normalized solution of (2.5) is

$$\varphi = \Lambda^{-1/2}\left[A(\log 1/\Lambda t)^{6\beta\xi} + B\Lambda t(\log 1/\Lambda t)^{-\beta(6\xi-3)}\right]\left[1 + O((\log 1/\Lambda t)^{-1})\right], \tag{2.33}$$

with A, B as in Ex.2(c). For $\xi = 0$, 1/6 the O-term has to be replaced by $O(t^2(\log 1/\Lambda t)^{\gamma})$.

The energy is

$$E \sim \frac{1}{2}\hbar\Lambda \frac{1}{(\Lambda t)^2} (\log 1/\Lambda t)^{-3\beta+12\beta\xi-2} |A|^2 \beta^2 6\xi(1-6\xi) \quad , \tag{2.34}$$

a limit case of (2.25) for $\alpha \to 0$.

For $\xi = 0$ we have instead of (2.34)

$$E \sim \frac{1}{2}\hbar\Lambda (\log 1/\Lambda t)^{3\beta} |B|^2 \quad , \tag{2.35}$$

and for $\xi=1/6$

$$E \sim \frac{1}{2}\hbar\Lambda (\log 1/\Lambda t)^{\beta} (|B|^2 + (\lambda-1) |A|^2) \quad . \tag{2.36}$$

In (2.36) we have to assume $\lambda \geq 1$, i. e. to exclude the ground state value $\lambda = \delta(2-\delta)$, because of the positivity condition in (2.10).

Example 4 the case of finite initial radius, cf.[12],

$$a(t) = b + c(\Lambda t)^{\alpha} \ , \alpha, b, c > 0 \ . \tag{2.37}$$

Energy and pressure are positive for $\alpha < 1$, and still singular,

$$\varepsilon \sim \frac{3\Lambda^2 c^2 \alpha^2}{b^2} (\Lambda t)^{2\alpha-2} \ , \ p \sim \frac{2\Lambda^2 c\alpha(1-\alpha)}{b} (\Lambda t)^{\alpha-2} \quad . \tag{2.38}$$

The solution of (2.5) is

$$\varphi = \Lambda^{-1/2} b^{-3/2} A\left[1 - 6\xi\frac{c}{b} (\Lambda t)^{\alpha} + O(t^{2\alpha})\right] + \Lambda^{-1/2} b^{-3/2} B\Lambda t\left[1 + O(t^{\alpha})\right] \quad , \tag{2.39}$$

with A, B as in Ex.2(c). For E we have

$$E \sim \frac{1}{2}\hbar\Lambda(\Lambda t)^{2\alpha-2} |A|^2 \alpha^2 \frac{c^2}{b^2} 6\xi(1-6\xi) \quad , \tag{2.40}$$

special treatment is again needed for $\xi = 0, 1/6$.

(a) $\xi = 1/6$

If we drop in (2.5, 2.7) the m^2 and a^{-2} terms that contribute only positive powers we can solve (2.5) by

$$\varphi = \Lambda^{-1/2} A a^{-1}(t) + \Lambda^{1/2} B a^{-1}(t) \int^t a^{-1}(t) dt \tag{2.41}$$

with A, B as before. E approaches a finite value,

$$E \sim \frac{1}{2}\hbar\Lambda b^{-1}\left[|B|^2 + |A|^2 (b^2 m^2 \Lambda^{-2} + \lambda - 1)\right] \quad . \tag{2.42}$$

(b) $\xi = 0$

With the same approximations as in (a) we have as independent solutions $1, \int a^{-3} dt$.

The normalized asymptotic solution of (2.5) is

$$\varphi = \Lambda^{-1/2} A b^{-3/2} + \Lambda^{1/2} t B b^{-3/2} [1 + O(t^{\alpha})] , \tag{2.43}$$

with the energy

$$E \sim \frac{1}{2} \hbar \Lambda \left[|B|^2 + |A|^2 (m^2 \Lambda^{-2} + \lambda b^{-2}) \right] , \tag{2.44}$$

which goes over for b=1 and $\varphi = \omega^{-1/2}(1 - i\omega t)$ in Eq. (47) of Ref.[14], likewise (2.42).

Example 5 a limit case of Ex.4,

$$a(t) = b + c(\log 1/\Lambda t)^{-\alpha} , \alpha, b, c > 0. \tag{2.45}$$

We have here always a positive ε and p,

$$\varepsilon \sim \frac{3c^2\alpha^2}{b^2} \frac{1}{t^2} (\log 1/\Lambda t)^{-2\alpha-2} , p \sim \frac{2c\alpha}{b} \frac{1}{t^2} (\log 1/\Lambda t)^{-\alpha-1} , \tag{2.46}$$

otherwise this case is completely analogous to Ex.4, if we replace in the formulae their $(\Lambda t)^{\alpha}$ by $(\log 1/\Lambda t)^{-\alpha}$. For the energy we obtain instead of (2.40)

$$E \sim \frac{1}{2} \hbar \Lambda \frac{1}{(\Lambda t)^2} (\log 1/\Lambda t)^{-2\alpha-2} |A|^2 \frac{\alpha^2 c^2}{b^2} 6\xi(1 - 6\xi) ; \tag{2.47}$$

formulae (2.42, 2.44) hold still true.

Example 6 a limit case of Ex.4,

$$a(t) = b + c\Lambda t(\log 1/\Lambda t)^{\alpha}, \alpha > 0. \tag{2.48}$$

ε and p are positive,

$$\varepsilon \sim \frac{3c^2\Lambda^2}{b^2} (\log 1/\Lambda t)^{2\alpha} , p \sim \frac{2c\alpha}{b} \frac{\Lambda}{t} (\log 1/\Lambda t)^{\alpha-1} . \tag{2.49}$$

If ξ does not take its two limit values 0, 1/6, we have as a fundamental system of (2.5) t , $1 - 6\xi c b^{-1} \Lambda t (\log 1/\Lambda t)^{\alpha}$, with

$$E \sim \frac{1}{2} \hbar \Lambda (\log 1/\Lambda t)^{2\alpha} |A|^2 \frac{c^2}{b^2} 6\xi(1-6\xi) . \tag{2.50}$$

The cases $\xi = 0, 1/6$ reduce to the preceding ones.

Example 7

Finally we discuss

$$a(t) = (\Lambda t)^{\alpha}, \alpha > 1, t \to 0. \tag{2.51}$$

These expansion factors violate the positivity of ε, but we will also see why that happens, and thus we think it is worthwhile to treat them here.

The normalized solution of (2.5) reads

$$\varphi = \Lambda^{-1/2}\cosh(r)\, e^{i\vartheta/2}\, \tilde{\varphi} + \Lambda^{-1/2}\sinh(r)\, e^{-i\vartheta/2}\, \tilde{\tilde{\varphi}} , \tag{2.52}$$

with

$$\tilde{\varphi} = B^{-1/4} \exp\left(i \frac{\sqrt{B}}{\alpha-1} (\Lambda t)^{1-\alpha}\right) (\Lambda t)^{-\alpha} \Big\{ 1 + iC(\Lambda t)^{\alpha-1} + O(t^{2(\alpha-1)}) +$$

$$+ \, im^2\Lambda^{-2}(\Lambda t)^{1+\alpha} D\left[1 + O(t^{\alpha-1})\right] + O(m^4 t^{2(1+\alpha)}) \Big\}, \tag{2.53}$$

and

$$B = \lambda - 6\xi \; , \; C = \frac{\alpha}{2}\frac{(1 - 2\alpha)(1 - 6\xi)}{\sqrt{B}\,(1 - \alpha)} \; , \; D = \frac{-1}{2(\alpha+1)\sqrt{B}} \; .$$

In (2.52) r and ϑ are parameters analogous to a, b, φ in (2.24).

The energy splits in

$$E = E_{monotonic} + E_{periodic} \; , \tag{2.54}$$

with

$$E_m = (\cosh^2(r) + \sinh^2(r))\, \hbar\Lambda(\Lambda t)^{-\alpha} \sqrt{B} + O(t^{\alpha-2}) \; , \tag{2.55}$$

$$E_p = -2\sinh(r)\cosh(r)\hbar\, t^{-1}\, \alpha(1-6\xi)\sin\left(2\sqrt{B}(\alpha-1)^{-1}(\Lambda t)^{1-\alpha} + \vartheta\right) + O(t^{-2+\alpha}) \; . \tag{2.56}$$

The case $\xi = 1/6$ needs again special treatment, instead of (2.53) we have

$$\tilde{\varphi} = B^{-1/4} \exp\left(i \frac{\sqrt{B}}{\alpha-1} (\Lambda t)^{1-\alpha}\right) (\Lambda t)^{-\alpha} \Big\{ 1 + m^2\Lambda^{-2}(\Lambda t)^{1+\alpha}\left[iD + E(\Lambda t)^{\alpha-1} + \right.$$

$$\left. + \, iF(\Lambda t)^{2(\alpha-1)} + O(t^{3(\alpha-1)})\right] + \left(m^2\Lambda^{-2}(\Lambda t)^{1+\alpha}\right)^2 G\left[1 + O(t^{\alpha-1})\right] + O(m^6\, t^{3(1+\alpha)}) \Big\}$$

with (2.57)

$$E = \frac{-1}{4B} \; , \; G = \frac{-1}{8B(\alpha+1)^2} \; .$$

F does not enter in the following equations that replace (2.55, 2.56),

$$E_m = (\cosh^2(r) + \sinh^2(r))\, \hbar\Lambda \left[\frac{\sqrt{B}}{(\Lambda t)^{\alpha}} + \frac{(\Lambda t)^{\alpha}\, m^2\Lambda^{-2}}{2\sqrt{B}} \right] + O(t^{3\alpha-2}) \; , \tag{2.58}$$

and

$$E_p = -\sinh(r)\cosh(r)\, \hbar\Lambda\, (\Lambda t)^{2\alpha-1} \frac{m^2\Lambda^{-2}\alpha}{B} \sin\left(2\frac{\sqrt{B}}{\alpha-1}(\Lambda t)^{1-\alpha} + \vartheta\right) + O(t^{3\alpha-2}) \; . \tag{2.59}$$

For $m = 0$ E_p is identically zero.

The frequency of an oscillation $\exp\left(\pm i\beta(\Lambda t)^{1-\alpha}\right)$, $t \to 0$, $\alpha > 1$, $\beta > 0$ as it occurs in (2.53, 2.57) and (2.56, 2.59) is asymptotically

$$\nu = 1/\Delta t \sim \frac{\Lambda\beta(\alpha - 1)}{2\pi(\Lambda t)^{\alpha}} \to \infty \; , \tag{2.60}$$

where Δt is determined by $\beta\left(\Lambda(t-\Delta t)\right)^{1-\alpha} - \beta(\Lambda t)^{1-\alpha} = 2\pi$. For E_m in (2.55, 2.58) we have then with $\beta = B^{1/2}(\alpha - 1)^{-1}$,

$$E_m \sim (\cosh^2(r) + \sinh^2(r))\, h\nu \quad . \tag{2.61}$$

E in (2.54) oscillates with a frequency of 2ν between the two curves

$$E_m(t) \pm 2\sinh(r)\cosh(r)\,\hbar t^{-1}\,\alpha(1-6\xi), \tag{2.62}$$

and for $\xi = 1/6$ between

$$E_m(t) \pm \sinh(r)\cosh(r)\,\hbar\Lambda\,(\Lambda t)^{2\alpha-1} m^2 \Lambda^{-2} \alpha/B \quad . \tag{2.63}$$

The break-down of formulae (2.1) based on classical relativistic hydrodynamics is not surprising in a situation when energy density and pressure diverge to infinity, and phenomena like (2.61) get dominant. Thus the conditions $\varepsilon > 0$, $p > 0$, with ε and p as in (2.1) are unlikely to give a good selection criterion for expansion factors in the limit $a(t) \to 0$. Thus we have decided to give in Examples 2 and 7 a discussion of factors $a(t) \sim (\Lambda t)^\alpha$ for the whole range of values $0 < \alpha < \infty$, and their bearing on the solutions of the wave equation.

III. DISCUSSION AND CONCLUSION

We outline at first the classical dynamics, namely geodesic motion on the space-time manifold, and compare then with the remnants of classical chaos in the energy formulae derived in Section 2.

Applying the geodesic variational principle to the line element ds^2 (Sec.1) on the covering space $R^+ \times B^3$, cf.[10], we calculate readily the geometric shapes of the geodesics,

$$r^2 + 2Mr\cos\varphi + (c/\Lambda)^2 = 0\,, \tag{3.1}$$

arcs of circles centred at $|\vec{M}| = M$, orthogonal to the boundary S_∞ of B^3 (c/Λ is the radius of B^3). Their time parametrization is given by

$$r^2(t) = (c/\Lambda)^2 \frac{\eta^2 + 1/4 - \eta\sqrt{1-(c/\Lambda M)^2}}{\eta^2 + 1/4 + \eta\sqrt{1-(c/\Lambda M)^2}}\,, \quad \eta(t) = C\exp\left(\pm\Lambda \int_{1/\Lambda}^{t} \frac{dt}{\sqrt{1+\mu^{-2}a^2(t)}\; a(t)}\right), \tag{3.2}$$

C and μ are integration constants, μ determines the hyperbolic length, possibly infinite, of the arc that is run through during the whole evolution $0 \le t \le \infty$ .The constant C fixes the location of the arc on (3.1). The parameter μ regulates the velocity via

$$\mu/a(t) = \frac{v/c}{\sqrt{1 - v^2/c^2}}\,, \tag{3.3}$$

for the definition of v and the derivation of (3.2, 3.3) see [14]. If $\mu = 0$ the particle is at rest, $\eta(t=0) = \eta(t=\infty) = C$; if $\mu = \infty$ then (3.2) gives the time parametrization of light rays, (3.1) holds also true for rays. For $\eta \to 0$ or $\eta \to \infty$ we have $r(t) \to c/\Lambda$, the trajectory approaches the boundary S_∞. This has interesting consequences for the chaotic properties of geodesic motion in the polyhedron F, as we will see. From the positivity condition $\varepsilon, p > 0$

in Ex.1, we know that $a(t) \sim \Lambda t$ for $t \to \infty$. With this asymptotic behaviour of a(t) in (3.2) we have $\eta(t = \infty, \mu)$ finite for $\mu < \infty$, and $\eta(t = \infty, \mu \to \infty) \sim \text{const.}\mu^{\pm 1}$, approaching 0 or ∞ for $v \to c$ according to (3.3).

Finally we discuss the behaviour of η for $t \to 0$. With $a(t) \sim (\Lambda t)^{\alpha}$, $0 < \alpha < 1$ as in Ex.2, we see easily from (3.2) that $\eta(t = 0, \mu)$ is finite and uniformly bounded away from 0 and ∞ for all μ. The same holds true for Exs.3-6. But for $\alpha \geq 1$ as in Ex.7 we have $\eta(t = 0, \mu) \to 0$ or ∞ regardless of the value of μ.

Up to now we have discussed geodesic motion in $R^+ \times B^3$, the covering manifold of our space-time manifold $R^+ \times F$, F is the polyhedron in B^3 that represents with its face-identification via Γ (cf. Sec.1) a hyperbolic 3-manifold, a spacelike section at a given instant of time. The concept of the covering space is the convenient tool to analyse the possibly very chaotic motion in F in simple terms. Every trajectory in F is constructed from an arc of a B^3-geodesic (3.1).This arc intersects a certain number of tiles $\gamma(F)$ of the tessellation $\Gamma(F)$, (cf. Sec.1 and [13, 15]). An arc piece lying in $\gamma(F)$ is projected via γ^{-1} into F. The trajectory in F consists thus of a number of arc pieces, whose initial and end points are identified by the face-pairing transformations of F, to give a smooth curve in F. The time parametrization of the B^3-geodesic is inherited by the F-geodesic. In this way a trajectory is realized in $R^+ \times F$.

The ergodic properties of the trajectory in F depend of course on the arc that is projected. To discuss them we cut the arc into two pieces, say $1/\Lambda \leq t \leq \infty$, and $0 \leq t \leq 1/\Lambda$, and consider the two limits $t \to \infty$, and $t \to 0$ separately. The initial point $t = 1/\Lambda$ lies always in B^3, the end point $t = \infty$ or $t = 0$ lies either in B^3 or on its boundary S_∞, depending on the value of $\eta(\infty)$ or $\eta(0)$.

If the end point lies inside B^3 the trajectory is bounded, i.e. lies inside a sphere of finite hyperbolic radius. Moreover, because the accumulation points of the tiling, the limit set $\Lambda(\Gamma)$ in Figs.1,2 lie on S_∞, the arc intersects only finitely many polyhedra, and thus there are only finitely many arc pieces in F constituting the trajectory. Its evolution is perfectly predictable and stable, because of its finite hyperbolic length.

If the end point lies on S_∞ , $\eta = 0$ or ∞ , there are two cases to distinguish. If it lies outside the limit set the trajectory intersects again only finitely many polyhedra, but clearly its F-projection is now unbounded reaching at $t = \infty$ or $t = 0$ a boundary of F on S_∞. There is also a positive Lyapounov exponent, but the propagation of the error in the initial conditions is only proportional to the hyperbolic distance that is run through. If the end point lies in $\Lambda(\Gamma)$, the arc intersects infinitely many tiles and that gives rise to chaotic behaviour of its projection. The F-trajectory is bounded, lying in a finite compact domain $C(\Lambda)\backslash\Gamma$ of 3-space, namely the intersection of the hyperbolic convex hull of the limit set with F, cf.[15]. It is mixing there and even Bernoullian.

Finally there is the case that the end point of the arc to be projected lies in B^3, but that its prolongation terminates in $\Lambda(\Gamma)$. We have than a finite arc on a trajectory whose F-projection is chaotic, as it may happen with massive particles. By increasing their speed, i.e. by increasing the chaoticity parameter μ in (3.2, 3.3), the end point can come arbitrarily close to S_∞, and the corresponding F-trajectory, though always regular can approximate its infinite and mixing prolongation to any wished degree.

To summarize, in the limit $t \to \infty$ there is a finite compact region $C(\Lambda)\backslash\Gamma$ in infinite 3-space in which chaotic motion can occur: rays have the Bernoulli property, and massive particles can approximate chaotic motion for $v \to c$ arbitrarily well in the above described way. A trajectory can enter this domain $C(\Lambda)\backslash\Gamma$ and it may get trapped there, but it can also go through unaffectedly, depending on its lifts into the covering space B^3, if they end in $\Lambda(\Gamma)$ or not. Massive particles are always bounded, whereas rays are either Bernoullian or unbounded.

The limit $t \to 0$: in the case of finite initial radius, Exs.4-6, particles and rays start to spread out regularly from inside the manifold, which already exists at $t = 0$ with a well defined metric and topology. In Exs.2,3,7 the 3-space contracts to a point, the distance between two points in the polyhedron F goes to zero. Keeping this in mind one can have nevertheless very different qualitative behaviour. Trajectories and rays in Exs.2,3 are regular and bounded for $t \to 0$, in Ex.7 they are either unbounded and regular or bounded and Bernoullian, trapped in $C(\Lambda)\backslash\Gamma$.

The time dependence of the energy in Eq. (2.11) and the frequency in (2.15) is remarkably similar to that of a classical particle moving along a geodesic,

$$E = mc^2 \sqrt{1 + \mu^2 a^{-2}(t)} = h\nu \tag{3.4}$$

with μ as in (3.3). For the wave length λ we have, using de Broglie's relation as above, $\lambda/a(t) = h/mc\mu$, cf.[14].

Concerning (2.12) we have classically the same time dependence, but there is a gap $\Delta\lambda = 1 - \delta(2 - \delta)$, δ the Hausdorff dimension of the limit set $\Lambda(\Gamma)$, in the spectrum of the L.-B. operator of the spacelike slices between the ground state wave function and the wave fields of the continuous spectrum.

From the eikonal , cf.[14],

$$\psi(t,r,\varphi) = -\,\omega \int_{1/A}^{t} a^{-1}(t)\, dt + \tilde{\psi}(r,\varphi) \tag{3.5}$$

we derive easily $\nu = c/\lambda \sim t^{-1} \left(e^{2\pi\Lambda/\omega} - 1\right)^{-1}$. This and the Einstein relation is again reflected in (2.15),(2.16). The ground state is excluded for massless particles, because the spectral parameter in (2.15) must be larger than 1.

The exponents in Exs.2,3 determining the singular behaviour of φ and E depend only on the exponent in a(t) and on ξ, the geometric coupling to the curvature scalar, neither the mass nor the spectral parameter enter in them. ξ ranges in the interval [1, 1/6], cf.(2.10), the limits $\xi = 0$, 1/6 are discontinuous in the power laws. There is no periodicity of φ and E in the limit $t \to 0$. The same holds true for Exs.4-6 (finite initial radius), in particular the classical ε in (2.38) and E in (2.40), have the same singular power laws, determined solely by the exponent in a(t). Only the limits $\xi = 0$, 1/6 have a finite energy.

Finally we discuss Ex.7. The energy in (2.55) and (2.58) has the same asymptotic behaviour as the classical E in (3.4) (which is not the case in Exs.2-6, except for $\xi = 0$, 1/6).The solution of the wave equation in (2.52) is periodic, and in (2.60), (2.61) we have the proportionality of E and ν like in (3.4) for $m > 0$. For rays we get from the eikonal in

(3.5) (with $a(t) \sim (\Lambda t)^{\alpha}$, $\alpha > 1$), and Einstein's relation the same time dependence as in the massive case, $E \sim (\Lambda t)^{-\alpha}\hbar\omega$.

In (2.60) the spectral variable λ (it should not be mixed up with the time dependent wave lengths in this section, we use the same notation) enters in the frequency, as it does in Ex.1 for m = 0, and the positivity condition in (2.10) imposes again restrictions on λ and ξ. For $m > 0$ and $6\xi < \delta(2-\delta)$ there is a gap in the frequency (2.60) increasing in time for $t \to 0$, corresponding to the gap $\Delta\lambda$ in the spectrum already observed in the limit $t \to \infty$ in Ex.1.

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REFERENCES

[1] Ahlfors L.V. (1981) Möbius Transformations in Several Dimensions, Lecture Notes (Univ. of Minnesota).
[2] Bers L. (1970) On Boundaries of Teichmüller Spaces and on Kleinian Groups, Ann. Math. 91, 570-600.
[3] Elstrodt J., F. Grunewald and J. Mennicke(1983) Discontinuous Groups on Three-Dimensional Hyperbolic Space, Russ. Math. Surv. 38 , 137-168.
[4] Hejhal D.A. (1987) Boundary-Groups, Degenerating Riemann Surfaces, and Spectral Theory, Univ. of Minnesota preprint.
[5] Krushkal S.L., B.N. Apanasov and N.A. Grusevskii (1986) Kleinian Groups and Uniformization in Examples and Problems, Transl. of Math. Monographs vol. 62 (Amer. Math. Soc., Providence, R.I.).
[6] McVittie G.C. (1956) General Relativity and Cosmology (Chapman and Hall,London).
[7] Misner C.W., K.S.Thorne and J.A.Wheeler (1973) Gravitation (Freeman, N.Y.)
[8] Patterson S.J. (1987) Lectures on Limit Sets of Kleinian Groups, in : Analytical and Geometrical Aspects of Hyperbolic Space, London Math. Soc. Lecture Notes 111, 281-323, Ed. D.B.Epstein (Cambridge Univ. Press, London).
[9] Scott P.(1983) The Geometries of 3-manifolds, Bull. London Math. Soc. 15, 401-487.
[10] Singer I.M. and J. A. Thorpe (1976) Lecture Notes on Elementary Topology and Geometry (Springer, N. Y.).
[11] Thurston W. (1978) The Geometry of 3-manifolds, Lecture Notes, (Princeton Univ.)
[12] Tolman R. C. (1934) Relativity, Thermodynamics and Cosmology (Clarendon Press, Oxford).
[13] Tomaschitz R. (1989) On the Calculation of Quantum Mechanical Ground States from Classical Geodesic Motion on Certain Spaces of Constant Negative Curvature, Physica D 34, 42-89.
[14] Tomaschitz R. (1991) Relativistic Quantum Chaos in Robertson-Walker Cosmologies, J. Math. Phys. 32, 2571-2579.
[15] Tomaschitz R.(1992) An Application of Kleinian Groups in the Quantization of an Unstable Dynamical System , Int. J. Theoret. Phys. 31, no.2 (to appear).
[16] Tomaschitz R., Quantum Chaos in de Sitter Cosmologies, to be published in the Proceedings of a NATO Advanced Research Workshop on Quantum Chaos, held May 28-June 1 1991 in Copenhagen; Ed. P. Cvitanovic (NATO-ASI Series, Plenum, NY).