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Quantum statistics of superluminal radiation

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Abstract

A statistical quantization of superluminal (tachyon) radiation is introduced. The tiny tachyonic fine structure constant suggests to depart from the usual quantum field theoretic expansions, and to use more elementary methods such as detailed equilibrium balancing of emission and absorption rates. Instead of commencing with an operator interpretation of the wave function, we quantize the time-averaged energy functional and the energy-balance equation. This allows to use different statistics for different types of modes. Transversal superluminal modes are quantized in Bose statistics, longitudinal ones are turned into fermions, resulting in a positive definite Hamiltonian for the radiation field. We discuss the absorptive space structure underlying superluminal quanta and the energy dissipation related to it. This dissipation leads to an adiabatic time variation of the temperature in the bosonic and fermionic spectral functions, gray-body quasi-equilibrium distributions with a dispersion relation adapted to the negative mass-square of the tachyonic modes. The superluminal radiation field couples by minimal substitution to subluminal matter. Adiabatically damped Einstein coefficients are obtained by detailed balancing, as well as emission and absorption rates for tachyon radiation in hydrogenic systems, in particular the possibility of spontaneous emission of superluminal fermionic quanta is pointed out, and time scales for the approach to equilibrium are derived.

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1. Introduction

Superluminal radiation fields have never been properly quantized, despite of various attempts, most notably in Refs. [1–3]. Here we carry out a purely statistical quantization, suggested by the very weak coupling of superluminal radiation to subluminal matter, some eleven orders below the electric fine structure constant. An elementary approach, avoiding propagators and field theoretical expansions, is quite sufficient to describe superluminal quantum effects such as the spontaneous emission of tachyons. The interaction of tachyonic quanta with matter will be settled by detailed balancing of emission and absorption rates, along Einstein's arguments, and the statistics of the superluminal modes will be determined by positivity requirements on energy.

The field theoretical second quantization of superluminal radiation has failed for two reasons. In a relativistic context, there is an insurmountable causality problem [4–12], and one also quickly runs into a positivity problem with regard to energy, accompanied by a variety of inconsistencies, such as the violation of unitarity and unstable ground states. In this paper, we focus on the statistics of modes rather than fields, on spectral distributions rather than propagators, and interactions are settled by equilibrium balancing rather than renormalization. Instead of prescribing commutators or anticommutators for the wave function, our starting point will be the operator interpretation of the time-averaged energy functional. The algebraic relations between the Fourier modes are chosen in a way to turn the indefinite classical energy functional into a positive definite Hamiltonian. This is achieved by using commutators for transversal modes, and anticommutators for longitudinal ones. The transversal modes resemble photons with a negative mass-square, but the third degree of freedom is subjected to the exclusion principle, the longitudinal modes being fermions.

The crucial point is to use Bose as well as Fermi statistics for one and the same field. This is in fact very unusual, as second quantization in three space dimensions is customarily set up by an operator interpretation of the classical wave function. The statistics, either Bose or Fermi, is then determined by the spin-statistics theorem [13], which, however, only applies to subluminal fields. The quantization here carried out turns the superluminal radiation field into a mixture of transversal bosons and longitudinal fermions. The tachyonic spectral functions are gray-body Fermi and Bose distributions in quasi-equilibrium (due to the energy dissipation discussed below), with a negative mass-square in the dispersion relation. For equilibrium to be reached, one has to admit emission and absorption processes, so that the particle numbers cannot be prescribed, and accordingly there is no chemical potential, even not for the fermionic modes. Superluminal quanta couple by minimal substitution to subluminal matter, and transition rates will be obtained by detailed balancing.

Tachyon radiation cannot be understood in a relativistic context, because of causality violation [12], it requires an absolute space, tantamount to a medium of wave propagation. Hence, when discussing statistical distributions of superluminal quanta, we have to reckon with the substance of space, the ether [15]. The absorptivity of the ether results in energy dissipation affecting the spectral functions. Superluminal radiation is thus gray- rather than black-body, provided the dissipation is sufficiently adiabatic for equilibrium mechanics to be applicable. This depends on the permeability of the ether and can be made quite quantitative. The derivation of the energy distributions of the bosonic transversal modes and the longitudinal fermions is centered at the time-averaged energy-balance equation, a Poynting theorem in Fourier space, relating the field energy and the energy flux to the dissipated energy. The indefinite classical energy functionals (of field energy and dissipated energy) become positive definite operators by quantizing transversal and longitudinal modes in different statistics as mentioned. In the spectral distributions, the refractivity of the ether enters by the dispersion relation, and its absorptivity generates an adiabatic time variation of the temperature. As an example, we will discuss the cosmic tachyon radiation, in particular how equilibrium can be reached by interactions with subluminal matter.

In Section 2, the tachyonic counterpart to Maxwell's equations is introduced and all that goes with it, such as tachyonic field strengths, inductions, permeabilities and material equations, and the time-averaging of the superluminal energy flux is discussed. In Section 3, we study the energy-balance and the dissipation effected by the absorptivity of the ether. The statistical interpretation of the energy-balance equation is given in Section 4 for transversal and in Section 5 for longitudinal modes. We derive the spectral energy densities, the Einstein coefficients, the emission and absorption rates for bosonic and fermionic tachyons, as well as tachyonic ionization cross sections. In Section 6, we present our conclusions. In the appendix, we calculate the thermodynamic variables and the equations of state for the fermionic modes.

2. Tachyons in a permeable spacetime

Tachyons emerge as an extension of the photon concept, a sort of photons with negative mass-square [15,16], satisfying the Maxwell equations

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{rot}\mathbf{E} + c^{-1}\partial\mathbf{B}/\partial t = 0\,,$$

$$\operatorname{div}\mathbf{D} = \rho - c^{-1}m_{\mathrm{t}}^2 C_0, \quad \operatorname{rot}\mathbf{H} - c^{-1}\partial\mathbf{D}/\partial t = c^{-1}\mathbf{j} + m_{\mathrm{t}}^2\mathbf{C}\,. \tag{2.1}$$

Tachyonic $\mathbf{E}$ and $\mathbf{B}$-fields relate to the vector potential by $\mathbf{E} = c^{-1}(\nabla A_0 - \partial\mathbf{A}/t)$ and $\mathbf{B} = \operatorname{rot}\mathbf{A}$. The inductive potential $(C_0, \mathbf{C})$ enters via the tachyon mass and connects to the vector potential by

$$C_0(t) = \int_{-\infty}^{+\infty} \varepsilon(t')A_0(t-t')\,\mathrm{d}t', \qquad \mathbf{A}(t) = \int_{-\infty}^{+\infty} \mu(t')\,\mathbf{C}(t-t')\,\mathrm{d}t'\,. \tag{2.2}$$

This complements the familiar material relations

$$\mathbf{D}(t) = \int_{-\infty}^{+\infty} \varepsilon(t')\,\mathbf{E}(t-t')\,\mathrm{d}t', \quad \mathbf{B}(t) = \int_{-\infty}^{+\infty} \mu(t')\,\mathbf{H}(t-t')\,\mathrm{d}t'\,, \tag{2.3}$$

where $\varepsilon(t)$ and $\mu(t)$ denote the dielectric and magnetic permeabilities, respectively. The tachyon mass m_{t} has the dimension of an inverse length, and is meant as a shortcut for $m_{\mathrm{t}}c/\hbar$. The signs of the mass terms in (2.1) are chosen in a way that $m_{\mathrm{t}}^2 > 0$ is the negative mass-square. We find $m_{\mathrm{t}}/m_{\mathrm{e}} \approx 1/238$, estimated from Lamb shifts in hydrogenic systems [17,18]. The tachyon field couples by minimal substitution to subluminal matter. In the case of a classical subluminal particle, the charge density and the current in (2.1) read $\rho = q\delta(\mathbf{x}-\mathbf{x}(t))$ and $\mathbf{j}=q\mathbf{v}\delta(\mathbf{x}-\mathbf{x}(t))$, where q is the tachyonic charge carried by the particle; quantized interactions will be discussed in Sections 4 and 5. Tachyonic and electric fine structure constants relate as $q^2/(4\pi\hbar c) \approx 1.0 \times 10^{-13} \approx 0.66\alpha^6$, again an estimate from Lamb shifts. We will not consider electromagnetic fields, so that we can use the notation so suggestive in electrodynamics without the risk of confusion. The mass term breaks the gauge invariance, and the tachyon potential $(A_0, \mathbf{A})$ becomes observable, resulting in a new set of material equations and an induction $(C_0, \mathbf{C})$. In (2.2) and (2.3), one may substitute $\varepsilon(t) = \delta(t) + \kappa(t)$ (δ stands for the Dirac function) and $\mu(t) = \delta(t) + \chi(t)$, the susceptibilities $\kappa(t)$ and $\chi(t)$ are required to vanish identically for negative t, as the response of the medium cannot happen prior to its exposure to the field.

Defining the Poynting vector as $\mathbf{S} = c\mathbf{E} \times \mathbf{H} + m_t^2 A_0 \mathbf{C}$, we obtain from the Maxwell equations and the material relations

$$\operatorname{div}\mathbf{S} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - m_t^2\left(\frac{1}{c^2}A_0\frac{\partial C_0}{\partial t} + \mathbf{C}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) = -\mathbf{j}\cdot\mathbf{E}\,. \tag{2.4}$$

If ε and μ are constant and $\mathbf{j} = 0$, we may write $\operatorname{div}\mathbf{S} + \partial\rho_E/\partial t = 0$, with the indefinite density

$$\rho_E = \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}) - \tfrac{1}{2}m_t^2(c^{-2}A_0C_0 + \mathbf{C}\cdot\mathbf{A})\,. \tag{2.5}$$

The energy of the transversal and longitudinal modes can be extracted from (2.4) by time averaging. To this end, we turn to the plane wave decomposition of the spatial component of the vector potential,

$$\mathbf{A}(\mathbf{x},t) = L^{-3/2}\sum_{\mathbf{k}}(\hat{\mathbf{A}}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\boldsymbol{x} - \mathrm{i}\omega^* t) + \text{c.c.}), \quad \hat{\mathbf{A}}(\mathbf{k}) := \sum_{\lambda=1}^{3}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\hat{a}(\mathbf{k},\lambda)\,, \tag{2.6}$$

with $\mathbf{k} := 2\pi\mathbf{n}/L$. The summation is over integer lattice points $\mathbf{n}$ in R^3, corresponding to periodic boundary conditions, so that the $L^{-3/2}\exp(\mathrm{i}\mathbf{k}\boldsymbol{x})$ are orthogonal and complete in a box of size L. The damping of plane waves is described by a complex frequency $\omega = \omega_R + \mathrm{i}\omega_I$, with $\omega_I \geqslant 0$, and ω^* denotes the complex conjugate. The amplitudes $\hat{\mathbf{A}}(\mathbf{k})$ (which are not really Fourier due to the damping factor, though we will name them so) are composed with arbitrary real unit vectors $\boldsymbol{\varepsilon}_{\mathbf{k},1}$ and $\boldsymbol{\varepsilon}_{\mathbf{k},2}$ (linear polarization vectors) orthogonal to $\boldsymbol{\varepsilon}_{\mathbf{k},3} := \mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$, so that the $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$ constitute an orthonormal triad for every $\mathbf{n}$. The amplitudes $\hat{a}(\mathbf{k},\lambda)$ are complex numbers. In (2.6) we depart from the standard electromagnetic formalism in a permeable medium, assuming a real wave vector $\mathbf{k}$ and an exponential damping factor $\exp(-\omega_I t)$, generated by a complex frequency. The amplitudes of the time component of the real vector potential are defined as in (2.6) with $(\mathbf{A}, \hat{\mathbf{A}})$ replaced by $(A_0(\mathbf{x},t), \hat{A}_0(\mathbf{k}))$, and the same holds for the field strengths and inductions.

The Maxwell equations (2.1) (with $\rho = 0$, $\mathbf{j} = 0$) read in Fourier space

$$\mathbf{k}\cdot\hat{\mathbf{B}} = 0, \quad \mathbf{k}\times\hat{\mathbf{E}} - \omega^* c^{-1}\hat{\mathbf{B}} = 0\,,$$

$$\mathbf{k}\cdot\hat{\mathbf{E}} = \mathrm{i}c^{-1}m_t^2\hat{A}_0, \quad \mathbf{k}\times\hat{\mathbf{B}} + c^{-1}\omega^*\hat{\varepsilon}(\omega^*)\hat{\mu}(\omega^*)\hat{\mathbf{E}} = -\mathrm{i}m_t^2\hat{\mathbf{A}}\,. \tag{2.7}$$

The Fourier coefficients of the field strengths and the vector potential relate as $\hat{\mathbf{E}}(\mathbf{k}) = \mathrm{i}c^{-1}(\mathbf{k}\hat{A}_0(\mathbf{k}) + \omega^*\hat{\mathbf{A}}(\mathbf{k}))$ and $\hat{\mathbf{B}}(\mathbf{k}) = \mathrm{i}\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{k})$. The permeabilities read in Fourier space

$$\hat{\varepsilon}(\omega) = \int_{-\infty}^{+\infty} \varepsilon(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t, \quad \hat{\varepsilon}(\omega^*) = \hat{\varepsilon}^*(-\omega), \tag{2.8}$$

and the same for $\hat{\mu}(\omega)$ and $\mu(t)$. Clearly, for $(\hat{A}_0, \hat{\mathbf{A}})$ to be a solution of the field equations (2.7), the dispersion relation,

$$k^2 = c^{-2}\omega^{*2}\hat{\varepsilon}(\omega^*)\hat{\mu}(\omega^*) + m_{\mathrm{t}}^2, \tag{2.9}$$

(with real $k := |\mathbf{k}|$) as well as the Lorentz condition,

$$c^{-2}\omega^*\hat{\varepsilon}(\omega^*)\hat{\mu}(\omega^*)\hat{A}_0 + \mathbf{k}\cdot\hat{\mathbf{A}} = 0, \tag{2.10}$$

must hold, which is also sufficient. As k^2 in (2.9) is real, $\omega^*(k) = \omega_{\mathrm{R}} - \mathrm{i}\omega_{\mathrm{I}}$ satisfies

$$\omega^{*2}\hat{\varepsilon}(\omega^*)\,\hat{\mu}(\omega^*) = \omega^2\hat{\varepsilon}^*(\omega^*)\,\hat{\mu}^*(\omega^*), \tag{2.11}$$

$\hat{\varepsilon}^*(\omega^*)$ denotes the complex conjugate of $\hat{\varepsilon}(\omega^*)$. Equation (2.9) is to be solved for (complex) $\omega(k)$, and Eq. (2.10) defines the Fourier coefficients $\hat{A}_0(\mathbf{k})$ of the time component, once the $\hat{\mathbf{A}}(\mathbf{k})$ are chosen. As the solution $\omega(k)$ is not unique, a further summation in (2.6) may be necessary, so that the coefficients $\hat{\mathbf{A}}(\mathbf{k}, i)$ and $\hat{a}(\mathbf{k}, \lambda, i)$ depend on a further index i labeling the branches $\omega_i(k)$. However, we will consider permeabilities where this branching does not occur. In any case, the coefficients $\hat{\mathbf{A}}(\mathbf{k})$ can arbitrarily be prescribed for each branch, that is, the amplitudes $\hat{a}(\mathbf{k}, \lambda)$ in (2.6), and then the $\hat{A}_0(\mathbf{k})$ are determined by (2.10) and (2.9). The Fourier coefficients of the inductions $\mathbf{D}$, $\mathbf{H}$, $\mathbf{C}$ and C_0 are obtained from the material relations (2.2) and (2.3),

$$\hat{\mathbf{D}}(\mathbf{k}) = \hat{\varepsilon}(\omega^*)\hat{\mathbf{E}}(\mathbf{k}), \quad \hat{\mathbf{B}} = \hat{\mu}\hat{\mathbf{H}}, \quad \hat{\mathbf{A}} = \hat{\mu}\hat{\mathbf{C}}, \quad \hat{C}_0 = \hat{\varepsilon}\hat{A}_0, \tag{2.12}$$

the argument in the permeabilities is ω^*, unless stated otherwise.

The vector potential can be split into a transversal and longitudinal component, cf. (2.6) and (2.10),

$$\hat{\mathbf{A}}^{\mathrm{T}} := \sum_{\lambda=1,2} \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\hat{a}(\lambda), \quad \hat{A}_0^{\mathrm{T}} = 0, \quad \hat{\mathbf{A}}^{\mathrm{L}} := \mathbf{k}_0\hat{a}(3), \quad \hat{A}_0^{\mathrm{L}} = -\frac{c^2k\hat{a}(3)}{\omega^*\hat{\varepsilon}\hat{\mu}}, \tag{2.13}$$

this decomposition is unique, as there is no gauge freedom. (We will frequently drop the argument $\mathbf{k}$ and/or the polarization index λ in the Fourier coefficients; the same holds for the argument ω^* in the permeabilities.) The field strengths read accordingly

$$\hat{\mathbf{E}}^{\mathrm{T}} = \mathrm{i}\frac{\omega^*}{c}\hat{\mathbf{A}}^{\mathrm{T}}, \quad \hat{\mathbf{B}}^{\mathrm{T}} = \mathrm{i}\mathbf{k}\times\hat{\mathbf{A}}^{\mathrm{T}}, \quad \hat{\mathbf{E}}^{\mathrm{L}} = -\frac{\mathrm{i}m_{\mathrm{t}}^2c}{\omega^*\hat{\varepsilon}\hat{\mu}}\mathbf{k}_0\hat{a}(3), \quad \hat{\mathbf{B}}^{\mathrm{L}} = 0, \tag{2.14}$$

where we used the dispersion relation (2.9). The inductions are then obtained from (2.12). Both the transversal and longitudinal components independently satisfy the field equations (2.7), with arbitrarily prescribed amplitudes $\hat{a}(\mathbf{k}, \lambda)$, $\lambda = 1, 2, 3$.

We consider time averages over a period of $2\pi/\omega_{\mathrm{R}}$; on this scale the variation of $\exp(-\omega_{\mathrm{I}}t)$ is assumed adiabatic. It is also understood that $\omega_{\mathrm{R}} + \mathrm{i}\omega_{\mathrm{I}} = \omega(k)$ solves the dispersion relation (2.9). We denote by $\Psi(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, t)$ any of the fields $\mathbf{A}, A_0, \mathbf{E}$,

B, or the inductions (2.12), all defined by series of type (2.6). $\hat{\Psi}(\mathbf{k})$ and $\hat{\Phi}(\mathbf{k})$ are the corresponding Fourier coefficients, such as $\hat{\mathbf{A}}(\mathbf{k}), \hat{\mathbf{E}}(\mathbf{k})$, etc. The time average of the product $\int_{L^3} \Psi\Phi \, d\mathbf{x}$ is readily calculated as $\langle \int \Psi\Phi \rangle = \sum_{\mathbf{k}} e^{-2\omega_I t} \hat{\Psi}(\mathbf{k})\hat{\Phi}^*(\mathbf{k}) + \text{c.c.}$ The integral sign refers to the spatial integration, and the averaging has been carried out mode by mode in the product series of $\int \Psi\Phi \, d\mathbf{x}$, cf. (2.6), and has as effect that terms containing $\hat{\Psi}(\mathbf{k})\hat{\Phi}(-\mathbf{k})$ and $\hat{\Psi}^*(\mathbf{k})\hat{\Phi}^*(-\mathbf{k})$ drop out. The damping factor $\exp(-\omega_I t)$ is regarded as constant within the averaging period $2\pi/\omega_R$. For instance,

$$\left\langle \int \mathbf{E}^2 \right\rangle = 2 \sum_{\mathbf{k}} e^{-2\omega_I t} \hat{\mathbf{E}}(\mathbf{k})\hat{\mathbf{E}}^*(\mathbf{k}), \quad \hat{\mathbf{E}}\hat{\mathbf{E}}^* = \hat{\mathbf{E}}^{\mathrm{T}}\hat{\mathbf{E}}^{\mathrm{T}*} + \hat{\mathbf{E}}^{\mathrm{L}}\hat{\mathbf{E}}^{\mathrm{L}*}, \tag{2.15}$$

$$\hat{\mathbf{E}}^{\mathrm{T}}\hat{\mathbf{E}}^{\mathrm{T}*} = c^{-2}|\omega|^2 \sum_{\lambda=1,2} \hat{a}\hat{a}^*, \quad \hat{\mathbf{E}}^{\mathrm{L}}\hat{\mathbf{E}}^{\mathrm{L}*} = m_t^4 c^2 |\omega|^{-2} |\hat{\varepsilon}|^{-2} |\hat{\mu}|^{-2} \hat{a}(3)\hat{a}^*(3). \tag{2.16}$$

The same relations (2.15) hold for the averaged square of **B** ($\mathbf{B}^{\mathrm{L}}=0$) and the inductions, cf. (2.12)–(2.14), with

$$\hat{\mathbf{D}}^{\mathrm{T,L}}\hat{\mathbf{D}}^{\mathrm{T,L}*} = |\hat{\varepsilon}|^2 \hat{\mathbf{E}}^{\mathrm{T,L}}\hat{\mathbf{E}}^{\mathrm{T,L}*}, \quad \hat{\mathbf{B}}\hat{\mathbf{B}}^* = |\hat{\mu}|^2 \hat{\mathbf{H}}\hat{\mathbf{H}}^* = k^2 \sum_{\lambda=1,2} \hat{a}\hat{a}^*. \tag{2.17}$$

Relations of type (2.15) also apply to the vector potential and the induced potential, to their spatial components as well as A_0 and C_0, with (2.16) replaced by

$$\hat{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{A}}^{\mathrm{T}*} = \sum_{\lambda=1,2} \hat{a}\hat{a}^*, \quad \hat{\mathbf{A}}^{\mathrm{L}}\hat{\mathbf{A}}^{\mathrm{L}*} = \hat{a}(3)\hat{a}^*(3), \quad \hat{\mathbf{C}}^{\mathrm{T,L}}\hat{\mathbf{C}}^{\mathrm{T,L}*} = |\hat{\mu}|^{-2}\hat{\mathbf{A}}^{\mathrm{T,L}}\hat{\mathbf{A}}^{\mathrm{T,L}*},$$

$$\hat{A}_0\hat{A}_0^* = |\hat{\varepsilon}|^{-2}\hat{C}_0\hat{C}_0^* = c^4 k^2 |\omega|^{-2} |\hat{\varepsilon}|^{-2} |\hat{\mu}|^{-2} \hat{a}(3)\hat{a}^*(3). \tag{2.18}$$

The argument in the permeabilities is $\omega^* = \omega_R - i\omega_I$, and ω is a solution of the dispersion relation (2.9), so that (2.11) holds. As for the Poynting vector defined before (2.4), we find the spatially integrated and time averaged flux

$$\left\langle \int \mathbf{S} \right\rangle = \sum_{\mathbf{k}} e^{-2\omega_I t} \left(\frac{c}{\hat{\mu}^*} \hat{\mathbf{E}} \times \hat{\mathbf{B}}^* + \frac{m_t^2}{\hat{\mu}^*} \hat{A}_0 \hat{\mathbf{A}}^* + \text{c.c.} \right). \tag{2.19}$$

The crucial point here is that the interference term of the transversal and longitudinal field components, $c\mathbf{E}^{\mathrm{L}} \times \mathbf{H} + m_t^2 A_0 \mathbf{C}^{\mathrm{T}}$, has vanished in the averaging procedure, and we arrive at

$$\left\langle \int \mathbf{S} \right\rangle = \left\langle \int \mathbf{S}^{\mathrm{T}} \right\rangle + \left\langle \int \mathbf{S}^{\mathrm{L}} \right\rangle, \quad \mathbf{S}^{\mathrm{T}} := c\mathbf{E}^{\mathrm{T}} \times \mathbf{H}, \quad \mathbf{S}^{\mathrm{L}} := m_t^2 A_0 \mathbf{C}^{\mathrm{L}}. \tag{2.20}$$

The flow components can be made more explicit by (2.13) and (2.14),

$$\left\langle \int \mathbf{S}^{\mathrm{T}} \right\rangle = 2 \sum_{\mathbf{k}} \mathbf{k} e^{-2\omega_I t} \,\mathrm{Re}\, \frac{\omega}{\hat{\mu}(\omega^*)} \sum_{\lambda=1,2} \hat{a}(\mathbf{k},\lambda)\hat{a}^*(\mathbf{k},\lambda),$$

$$\left\langle \int \mathbf{S}^{\mathrm{L}} \right\rangle = -2 m_t^2 c^2 \sum_{\mathbf{k}} \mathbf{k} e^{-2\omega_I t} |\hat{\mu}(\omega^*)|^{-2} \,\mathrm{Re}\, \frac{1}{\omega \hat{\varepsilon}^*(\omega^*)} \hat{a}(\mathbf{k},3)\hat{a}^*(\mathbf{k},3). \tag{2.21}$$

The meaning of this decomposition into a transversal and longitudinal flux will get apparent in the next section, when we discuss the corresponding splitting of the energy density and the energy dissipation in the ether.

3. Energy balance for superluminal radiation in a refractive and absorptive space–time

To identify the energy density of the transversal and longitudinal modes as well as the dissipated energy, we derive at first the time average of the conservation law (2.4), with the external current dropped. We start with the $\mathbf{E}\cdot\partial\mathbf{D}/\partial t$ term in (2.4), write

$$\frac{\partial}{\partial t}(\hat{\mathbf{D}}(\mathbf{k})\exp(-\mathrm{i}\omega^* t)) = -\mathrm{i}\omega^*\hat{\varepsilon}(\omega^*)\mathrm{e}^{-\omega_\mathrm{I}t}\hat{\mathbf{E}}(\mathbf{k})\mathrm{e}^{-\mathrm{i}\omega_\mathrm{R}t}\,, \tag{3.1}$$

and use a standard expansion [19] in the imaginary part of the frequency $\omega^* := \omega_\mathrm{R} - \mathrm{i}\omega_\mathrm{I}$,

$$\omega^*\hat{\varepsilon}(\omega^*)\mathrm{e}^{-\omega_\mathrm{I}t} = \sum_{n=0}^{\infty}(\omega_\mathrm{R}\hat{\varepsilon}(\omega_\mathrm{R}))^{(n)}\frac{\mathrm{i}^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{-\omega_\mathrm{I}t}\,, \tag{3.2}$$

which implies adiabatic damping compared to the harmonic time variation. We so find, by means of the time averaging defined after (2.14),

$$\left\langle\int \mathbf{E}\cdot\partial\mathbf{D}/\partial t\right\rangle = \sum_{\mathbf{k}}\Big(2\omega_\mathrm{R}\,\mathrm{Im}\,\hat{\varepsilon}(\omega_\mathrm{R})\hat{\mathbf{E}}(\mathbf{k})\hat{\mathbf{E}}^*(\mathbf{k})\,\mathrm{e}^{-2\omega_\mathrm{I}t} + \mathrm{Re}(\omega_\mathrm{R}\hat{\varepsilon}(\omega_\mathrm{R}))'\frac{\partial}{\partial t}(\hat{\mathbf{E}}(\mathbf{k})\hat{\mathbf{E}}^*(\mathbf{k})\,\mathrm{e}^{-2\omega_\mathrm{I}t})\Big) + \mathrm{O}(\omega_\mathrm{I}^2)\,. \tag{3.3}$$

This also holds with the replacements $(\mathbf{E},\mathbf{D},\hat{\varepsilon})\to(\mathbf{H},\mathbf{B},\hat{\mu})$, $(\mathbf{C},\mathbf{A},\hat{\mu})$ and $(A_0,C_0,\hat{\varepsilon})$, respectively, and thus we may write the time average of (2.4) as

$$-\left\langle\int \mathrm{div}\,\mathbf{S}\right\rangle = \partial\left\langle\int\rho_\mathrm{E}\right\rangle\Big/\partial t + \left\langle\int I_\mathrm{dis}\right\rangle\,, \tag{3.4}$$

with the averaged energy density

$$\left\langle\int\rho_\mathrm{E}\right\rangle = \sum_{\mathbf{k}}\mathrm{e}^{-2\omega_\mathrm{I}t}\,\mathrm{Re}\,(\omega_\mathrm{R}\hat{\varepsilon}(\omega_\mathrm{R}))'(\hat{\mathbf{E}}\hat{\mathbf{E}}^* - m_\mathrm{t}^2c^{-2}\hat{A}_0\hat{A}_0^*) + \sum_{\mathbf{k}}\mathrm{e}^{-2\omega_\mathrm{I}t}\,\mathrm{Re}\,(\omega_\mathrm{R}\hat{\mu}(\omega_\mathrm{R}))'(\hat{\mathbf{H}}\hat{\mathbf{H}}^* - m_\mathrm{t}^2\hat{\mathbf{C}}\hat{\mathbf{C}}^*) + \mathrm{O}(\omega_\mathrm{I}^2)\,, \tag{3.5}$$

and the same for the dissipated energy per unit time, $\langle\int I_\mathrm{dis}\rangle$, but with $\mathrm{Re}\,(\omega_\mathrm{R}\hat{\varepsilon})'$ and $\mathrm{Re}\,(\omega_\mathrm{R}\hat{\mu})'$ replaced by $2\omega_\mathrm{R}\,\mathrm{Im}\,\hat{\varepsilon}(\omega_\mathrm{R})$ and $2\omega_\mathrm{R}\,\mathrm{Im}\,\hat{\mu}(\omega_\mathrm{R})$, respectively. We will identify $\langle\int\rho_\mathrm{E}\rangle$ with the field energy, and $\langle\int I_\mathrm{dis}\rangle$ with the dissipated energy, but to do so we still have to disentangle the transversal and longitudinal components like in (2.21).

The preceding time averages as well as the flux components (2.21) can be further simplified by performing the ω_I-expansion in the dispersion relation (2.9). To this end we assume $\mathrm{Im}(\hat{\varepsilon}(\omega^*),\hat{\varepsilon}',\hat{\mu},\hat{\mu}') = \mathrm{O}(\omega_\mathrm{I})$, so that

$$\mathrm{Re}\,\hat{\varepsilon}(\omega^*) = \mathrm{Re}\,\hat{\varepsilon}(\omega_\mathrm{R}) + \mathrm{O}(\omega_\mathrm{I}^2),$$
$$\mathrm{Im}\,\hat{\varepsilon}(\omega^*) = \mathrm{Im}\,\hat{\varepsilon}(\omega_\mathrm{R}) - \omega_\mathrm{I}\,\mathrm{Re}\,\hat{\varepsilon}'(\omega_\mathrm{R}) + \mathrm{O}(\omega_\mathrm{I}^2), \tag{3.6}$$

and the same expansions hold for $\hat{\mu}(\omega)$. By substituting this into (2.9), we easily obtain

$$k^2 = \omega_R^2 c^{-2} \operatorname{Re}\hat{\varepsilon}(\omega_R) \operatorname{Re}\hat{\mu}(\omega_R) + m_t^2 \,, \tag{3.7}$$

$$\omega_I = \omega_R \frac{\operatorname{Re}\hat{\varepsilon} \operatorname{Im}\hat{\mu} + \operatorname{Im}\hat{\varepsilon} \operatorname{Re}\hat{\mu}}{\operatorname{Re}(\omega_R\hat{\varepsilon})' \operatorname{Re}\hat{\mu} + \operatorname{Re}(\omega_R\hat{\mu})' \operatorname{Re}\hat{\varepsilon}} \,. \tag{3.8}$$

In (3.8) the argument of the permeabilities is ω_R, calculated via (3.7) as a function of k. The solution $\omega(k)=\omega_R+i\omega_I$ of (3.7) and (3.8) also solves the dispersion relation (2.9) up to terms of $O(\omega_I^2)$, and we see from (3.8) that our assumption $\operatorname{Im}(\hat{\varepsilon}, \hat{\varepsilon}', \hat{\mu}, \hat{\mu}')=O(\omega_I)$ is self-consistent. We may now write in (2.21)

$$\left\langle \int \mathbf{S}^{\mathrm{T}} \right\rangle = 2\sum_{\mathbf{k}} \mathbf{k} e^{-2\omega_I t} \frac{\omega_R}{\operatorname{Re}\hat{\mu}} \sum_{\lambda=1,2} \hat{a}\hat{a}^*,$$

$$\left\langle \int \mathbf{S}^{\mathrm{L}} \right\rangle = -2m_t^2 c^2 \sum_{\mathbf{k}} \mathbf{k} e^{-2\omega_I t} \frac{\hat{a}(3)\hat{a}^*(3)}{\omega_R \operatorname{Re}^2\hat{\mu} \operatorname{Re}\hat{\varepsilon}} \,. \tag{3.9}$$

up to $O(\omega_I^2)$. If not otherwise indicated, $\operatorname{Re}\hat{\varepsilon} := \operatorname{Re}(\hat{\varepsilon}(\omega_R))$, $\operatorname{Im}\hat{\varepsilon} := \operatorname{Im}(\hat{\varepsilon}(\omega_R))$, $\operatorname{Re}\hat{\varepsilon}' := \operatorname{Re}(\hat{\varepsilon}'(\omega_R))$, and the same for $\hat{\mu}$; the prime just means ordinary differentiation. If we write $\hat{\varepsilon}$ and $\hat{\mu}$ without explicit indication of its real or imaginary part, we always mean $\hat{\varepsilon}(\omega^*)$ and $\hat{\mu}(\omega^*)$, as in Section 2. Analyticity properties, Kramers–Kronig relations, and the fluctuation–dissipation theorem for tachyonic field strengths and inductions will not be discussed in this paper. It is also understood that $\omega_R(k)$ and $\omega_I(k)$ depend on the wave vector, being solutions of (3.7) and (3.8).

In the averages $\langle \int \rho_E \rangle$ and $\langle \int I_{\mathrm{dis}} \rangle$, cf. (3.5), we rewrite the products of the Fourier components by means of (2.16), (2.17), (2.9) and (2.11), and then carry out the ω_I-expansion as done in (3.6)–(3.9),

$$\hat{\mathbf{E}}\hat{\mathbf{E}}^* - m_t^2 c^{-2} \hat{A}_0 \hat{A}_0^* = \frac{\omega_R^2}{c^2} \sum_{\lambda=1,2} \hat{a}\hat{a}^* - \frac{m_t^2}{\operatorname{Re}\hat{\varepsilon} \operatorname{Re}\hat{\mu}} \hat{a}(3)\hat{a}^*(3) + O(\omega_I^2) \,,$$

$$\hat{\mathbf{H}}\hat{\mathbf{H}}^* - m_t^2 \hat{\mathbf{C}}\hat{\mathbf{C}}^* = \frac{\operatorname{Re}\hat{\varepsilon}}{\operatorname{Re}\hat{\mu}} (\hat{\mathbf{E}}\hat{\mathbf{E}}^* - m_t^2 c^{-2} \hat{A}_0 \hat{A}_0^*) + O(\omega_I^2). \tag{3.10}$$

The λ-summations stem from the transversal components $\hat{\mathbf{E}}^{\mathrm{T}}\hat{\mathbf{E}}^{\mathrm{T}*}$ and $\hat{\mathbf{H}}\hat{\mathbf{H}}^* - m_t^2 \hat{\mathbf{C}}^{\mathrm{T}}\hat{\mathbf{C}}^{\mathrm{T}*}$, respectively, and the $\hat{a}(3)\hat{a}^*(3)$ term can be identified with the longitudinal components $\hat{\mathbf{E}}^{\mathrm{L}}\hat{\mathbf{E}}^{\mathrm{L}*} - m_t^2 c^{-2} \hat{A}_0 \hat{A}_0^*$ and $-m_t^2 \hat{\mathbf{C}}^{\mathrm{L}}\hat{\mathbf{C}}^{\mathrm{L}*}$. In this way we can unambiguously separate the contributions of the transversal and longitudinal modes, $\langle \int \rho_E \rangle = \langle \int \rho_E^{\mathrm{T}} \rangle + \langle \int \rho_E^{\mathrm{L}} \rangle$ and $\langle \int I_{\mathrm{dis}} \rangle = \langle \int I_{\mathrm{dis}}^{\mathrm{T}} \rangle + \langle \int I_{\mathrm{dis}}^{\mathrm{L}} \rangle$, where the transversal averages are

$$\left\langle \int \rho_E^{\mathrm{T}} \right\rangle = \sum_{\mathbf{k}} \frac{\omega_R^2}{c^2} e^{-2\omega_I t} \left(\operatorname{Re}(\omega_R \hat{\varepsilon})' + \operatorname{Re}(\omega_R \hat{\mu})' \frac{\operatorname{Re}\hat{\varepsilon}}{\operatorname{Re}\hat{\mu}} \right) \sum_{\lambda=1,2} \hat{a}\hat{a}^* \,,$$

$$\left\langle \int I_{\mathrm{dis}}^{\mathrm{T}} \right\rangle = 2\sum_{\mathbf{k}} \omega_R \frac{\omega_R^2}{c^2} e^{-2\omega_I t} \left(\operatorname{Im}\hat{\varepsilon} + \operatorname{Im}\hat{\mu} \frac{\operatorname{Re}\hat{\varepsilon}}{\operatorname{Re}\hat{\mu}} \right) \sum_{\lambda=1,2} \hat{a}\hat{a}^* \,, \tag{3.11}$$

and the same holds for $\langle\int\rho_{\rm E}^{\rm L}\rangle$ and $\langle\int I_{\rm dis}^{\rm L}\rangle$, but with $\omega_{\rm R}^2/c^2$ replaced by $-m_{\rm t}^2/({\rm Re}\,\hat\varepsilon\,{\rm Re}\,\hat\mu)$, and the λ-sum over the transversal polarizations is replaced by $\hat a(3)\,\hat a^*(3)$. This is valid up to terms of ${\rm O}(\omega_{\rm I}^2)$, like the corresponding decomposition (3.9) of the flow, and the argument in the permeabilities is $\omega_{\rm R}$. Both $\omega_{\rm R}$ and $\omega_{\rm I}$ depend on the wave vector, being solutions of (3.7) and (3.8). Moreover, $\langle\int\rho_{\rm E}^{\rm T,L}\rangle={\rm O}(1)$, $\langle\int{\bf S}^{\rm T,L}\rangle={\rm O}(1)$, and $\langle\int I_{\rm dis}^{\rm T,L}\rangle={\rm O}(\omega_{\rm I})$.

The positivity of the energy averages $\langle\int\rho_{\rm E}^{\rm T}\rangle$ and $\langle\int I_{\rm dis}^{\rm T}\rangle$ is ensured by requiring

$$
{\rm Re}\,(\omega_{\rm R}\hat\varepsilon(\omega_{\rm R}))'>0,\quad {\rm Im}\,\hat\varepsilon\geqslant 0,\quad {\rm Re}\,\hat\varepsilon\,{\rm Re}\,\hat\mu>0\,, \tag{3.12}
$$

and the same with $\hat\varepsilon$ and $\hat\mu$ interchanged. Under the same conditions, the longitudinal components are negative definite. In Section 5, we will quantize them in Fermi statistics, effecting an overall sign change of $\langle\int\rho_{\rm E}^{\rm L}\rangle$, $\langle\int I_{\rm dis}^{\rm L}\rangle$, and $\langle\int{\bf S}^{\rm L}\rangle$. The transversal modes will be quantized in Bose statistics, which does not affect the positivity of the transversal energy. In this way we will obtain a positive definite Hamilton operator for the transversal as well as longitudinal modes, so that we can identify $\langle\int\rho_{\rm E}\rangle$ as field energy, and $\langle\int I_{\rm dis}\rangle$ stands for the energy per unit time dissipated into the ether. The conservation law (3.4) holds for the transversal and longitudinal components individually.

As a consistency check, we consider a single mode $\mathbf{k}$ in the series (3.9) and (3.11), and find, with ${\bf v}_{\rm gr}:={\bf k}_0\,{\rm d}\omega_{\rm R}/{\rm d}k$,

$$
\left\langle\int{\bf S}^{\rm T,L}\right\rangle_{\bf k}=\left\langle\int\rho_{\rm E}^{\rm T,L}\right\rangle_{\bf k}{\bf v}_{\rm gr},\quad \frac{{\rm d}\omega_{\rm R}}{{\rm d}k}=\frac{2c^2k}{\omega_{\rm R}({\rm Re}\,(\omega_{\rm R}\hat\varepsilon)'\,{\rm Re}\,\hat\mu+{\rm Re}\,(\omega_{\rm R}\hat\mu)'{\rm Re}\,\hat\varepsilon)}\,, \tag{3.13}
$$

where the group velocity is determined by the dispersion relation (3.7).

To prepare the second quantization in Sections 4 and 5, we introduce rescaled Fourier coefficients $a(\mathbf{k},\lambda)$ in the preceding time averages, so that $\hat a(\mathbf{k},\lambda)=:\alpha^{\rm T}a(\mathbf{k},\lambda)$ for $\lambda=1,2$, and $\hat a(\mathbf{k},3)=:\alpha^{\rm L}a(\mathbf{k},3)$, with the normalization factors

$$
\alpha^{\rm T}:=\frac{c\sqrt{\hbar{\rm Re}\,\hat\mu/\omega_{\rm R}}}{({\rm Re}\,(\omega_{\rm R}\hat\varepsilon)'{\rm Re}\,\hat\mu+{\rm Re}\,(\omega_{\rm R}\hat\mu)'\,{\rm Re}\,\hat\varepsilon)^{1/2}},\quad \alpha^{\rm L}:=\frac{\omega_{\rm R}}{m_{\rm t}c}\sqrt{{\rm Re}\,\hat\varepsilon\,{\rm Re}\,\hat\mu}\,\alpha^{\rm T}\,. \tag{3.14}
$$

The square of the transversal normalization factor relates to the group velocity in (3.13) via ${\rm d}\omega_{\rm R}/{\rm d}k=2k\alpha^{{\rm T}^2}/(\hbar{\rm Re}\,\hat\mu)$. We introduce the adiabatically varying frequency

$$
\tilde\omega(k,t):=\omega_{\rm R}(k)\exp(-2\omega_{\rm I}(k)t)\,, \tag{3.15}
$$

so that the energy density in (3.11) gets a familiar shape,

$$
\left\langle\int\rho_{\rm E}^{\rm T}\right\rangle=\hbar\sum_{\bf k}\tilde\omega\sum_{\lambda=1,2}a(\mathbf{k},\lambda)a^*(\mathbf{k},\lambda),\qquad \left\langle\int\rho_{\rm E}^{\rm L}\right\rangle=-\hbar\sum_{\bf k}\tilde\omega a(\mathbf{k},3)a^*(\mathbf{k},3)\,. \tag{3.16}
$$

The frequency $\tilde\omega(k,t)$ is determined by (3.7) and (3.8); conditions (3.12) also leave $\omega_{\rm I}$ positive (or zero, in the limit of real permeabilities). In the normalization (3.14), the averaged energy flux reads, cf. (3.9) and (3.13),

$$\left\langle \int \mathbf{S}^{\mathrm{T}} \right\rangle = \sum_{\mathbf{k}} \mathbf{v}_{\mathrm{gr}} \hbar \tilde{\omega} \sum_{\lambda=1,2} a(\mathbf{k},\lambda) a^*(\mathbf{k},\lambda)\,,$$

$$\left\langle \int \mathbf{S}^{\mathrm{L}} \right\rangle = -\sum_{\mathbf{k}} \mathbf{v}_{\mathrm{gr}} \hbar \tilde{\omega} a(\mathbf{k},3) a^*(\mathbf{k},3)\,, \tag{3.17}$$

valid up to terms of $\mathrm{O}(\omega_{\mathrm{I}}^2)$ like (3.16). The energy per unit time dissipated by transversal and longitudinal modes is found as, cf. (3.11) and (3.8),

$$\left\langle \int I_{\mathrm{dis}}^{\mathrm{T}} \right\rangle = 2\hbar \sum_{\mathbf{k}} \omega_{\mathrm{I}} \tilde{\omega} \sum_{\lambda=1,2} a(\mathbf{k},\lambda) a^*(\mathbf{k},\lambda)\,,$$

$$\left\langle \int I_{\mathrm{dis}}^{\mathrm{L}} \right\rangle = -2\hbar \sum_{\mathbf{k}} \omega_{\mathrm{I}} \tilde{\omega} a(\mathbf{k},3) a^*(\mathbf{k},3)\,, \tag{3.18}$$

again up to $\mathrm{O}(\omega_{\mathrm{I}}^2)$.

As for the microscopic structure of the ether, we consider a classical oscillator model [15], which gives $\hat{\mu} \approx 1$ and

$$\hat{\varepsilon}(\omega) \approx 1 + \alpha_0 g(\omega), \quad g(\omega) := (\omega_0^2 - \omega^2 - \mathrm{i}\gamma_0\omega)^{-1}\,, \tag{3.19}$$

with $\alpha_0 := N_0 q_0^2 m_0^{-1}$ and $\gamma_0 > 0$. Here, ω_0 is the free oscillator frequency, γ_0 the damping constant, N_0 the number density, m_0 the mass, and q_0 the tachyonic charge of the uniformly distributed oscillators constituting the ether. We assume a narrow line breadth, $\gamma_0 \to 0$, and real ω. The maximum of $\mathrm{Im}\,\hat{\varepsilon}$ is $\mathrm{Im}\,\hat{\varepsilon}(\omega_{\max}) \sim \alpha_0/(\gamma_0\omega_0)$, located at $\omega_{\max} \sim \omega_0$. In the vicinity of the resonance, we find the Lorentzian

$$\mathrm{Im}\,\hat{\varepsilon} \approx \frac{\alpha_0\gamma_0}{4\omega_0} \frac{1}{(\omega-\omega_0)^2 + (\gamma_0/2)^2}\,. \tag{3.20}$$

The real part of $\hat{\varepsilon}(\omega) - 1$ has apparently a zero at ω_0 and one or two extrema,

$$\omega_{\mathrm{min/max}} = \sqrt{\omega_0(\omega_0 \pm \gamma_0)}, \quad \mathrm{Re}\,\hat{\varepsilon}(\omega_{\mathrm{min/max}}) - 1 = \mp \frac{\alpha_0}{2\gamma_0} \frac{1}{\omega_0 \pm \gamma_0/2}\,. \tag{3.21}$$

The Drude formula (3.19) is meant as a power series expansion in α_0. As mentioned, $\hat{\mu}(\omega) \approx 1$, and thus the positivity conditions (3.12) are satisfied. For $\omega \to \infty$, we find $\mathrm{Re}\,\hat{\varepsilon} \sim \mathrm{Re}\,(\omega\hat{\varepsilon})' = 1 + \mathrm{O}(\omega^{-2})$ and $\mathrm{Im}\,\hat{\varepsilon} \sim \alpha_0\gamma_0\omega^{-3}$. If $\omega \to 0$, $\mathrm{Re}\,\hat{\varepsilon} \sim \mathrm{Re}\,(\omega\hat{\varepsilon})' = 1 + \alpha_0/\omega_0^2 + \mathrm{O}(\omega^2)$ and $\mathrm{Im}\,\hat{\varepsilon} \sim \alpha_0\gamma_0\omega_0^{-4}\omega$, so that in the respective limits, cf. (3.8),

$$\omega_{\mathrm{I}}(\omega_{\mathrm{R}} \to \infty) \sim \frac{\alpha_0\gamma_0}{2\omega_{\mathrm{R}}^2}, \quad \omega_{\mathrm{I}}(\omega_{\mathrm{R}} \to 0) \sim \frac{\alpha_0\gamma_0}{2(1+\alpha_0/\omega_0^2)\omega_0^4}\,\omega_{\mathrm{R}}^2\,. \tag{3.22}$$

Accordingly, $\omega_{\mathrm{I}}(\omega_{\mathrm{R}})$ is uniformly α_0-small in the whole frequency range. We will return to these limits when discussing tachyonic gray-body radiation, cf. (4.7), ω_{I} determines the adiabatic time variation of the temperature in the spectral distributions.

4. Transversal tachyons in Bose-Einstein statistics

We quantize in occupation number representation. The Fourier coefficients $a(\mathbf{k},\lambda)$ in the time averaged transversal energy density (3.16) are replaced by operators a_i, and their complex conjugates $a^*(\mathbf{k},\lambda)$ by the adjoints a_i^+. Bose statistics is defined by the commutation relations $[a_i,a_j^+]=\delta_{ij}$, $[a_i,a_j]=0$ and $[a_i^+,a_j^+]=0$; the indices i and j stand for the modes $(\mathbf{k},\lambda)$, $\mathbf{k}=2\pi\mathbf{n}/L$, $\mathbf{n}\in Z^3$, $\lambda=1,2$, as defined after (2.6). Orthogonal transversal states do not affect each other, and the longitudinal degree of freedom can likewise be treated independently, cf. Section 5, as operators in different orthogonal subspaces are supposed to commute. Therefore, to save notation, we will drop the polarization index λ, that is, quantize modes of a given linear polarization, $\lambda=1$, say.

The number operators $N_i:=a_i^+a_i$ are Hermitian and commute. We consider the basis vectors $|n_1,\ldots,n_i,\ldots,n_\infty\rangle$ (shortcut $|n\rangle$, e.g. $|0\rangle$ for the vacuum state). The occupation numbers n_i are non-negative integers indicating the number of particles in state i. In each basis vector, only a finite number of the n_i are non-zero. A scalar product is defined by $\langle n|n'\rangle=\delta_{n_1,n_1'}\ldots\delta_{n_\infty,n_\infty'}$. Operators satisfying the above commutation relations are readily found,

$$a_i|n_1,\ldots,n_i,\ldots,n_\infty\rangle=\sqrt{n_i}|n_1,\ldots,n_i-1,\ldots,n_\infty\rangle\,,$$

$$a_i^+|n_1,\ldots,n_i,\ldots,n_\infty\rangle=\sqrt{n_i+1}|n_1,\ldots,n_i+1,\ldots,n_\infty\rangle\,, \tag{4.1}$$

and $a_i|n\rangle=0$ (zero-vector) if $n_i=0$. Hence, $\langle a_i^+n|n'\rangle=\langle n|a_in'\rangle$, as well as $N_i|n\rangle=n_i|n\rangle$. We so find the Hamilton operator of the $\boldsymbol{\varepsilon}_{\mathbf{k},1}$-polarized modes in (3.16) as

$$\left\langle\int\rho_{\mathrm{E}}^{\mathrm{T1}}\right\rangle=\hbar\sum_{\mathbf{k}}\tilde{\omega}a^+(\mathbf{k},1)a(\mathbf{k},1)\,, \tag{4.2}$$

and the same for the $\boldsymbol{\varepsilon}_{\mathbf{k},2}$-modes. The partition function and the internal energy of modes of a given transversal polarization is calculated in the usual way,

$$\left\langle n\left|\int\rho_{\mathrm{E}}^{\mathrm{T1}}\right|n\right\rangle=E(n_1,\ldots,n_\infty)=\sum_{i=1}^{\infty}\hbar\tilde{\omega}_in_i\,,$$

$$Z=\sum_{n_1,\ldots,n_\infty=0}^{\infty}\exp(-\beta E(n_1,\ldots,n_\infty))=\prod_{\mathbf{k}}\sum_{l=0}^{\infty}\exp(-\beta\hbar\tilde{\omega}(|\mathbf{k}|,\lambda;t)l)\,, \tag{4.3}$$

$$\log Z=-\sum_{\mathbf{k}}\log(1-\exp(-\beta\hbar\tilde{\omega})),\quad U=\sum_{\mathbf{k}}\frac{\hbar\tilde{\omega}}{\exp(\beta\hbar\tilde{\omega})-1}\,. \tag{4.4}$$

In the thermodynamic limit [20], $\sum_{\mathbf{k}}\to L^3/(2\pi)^3\int d\mathbf{k}$, we find

$$\frac{U}{V}=\frac{1}{2\pi^2}\int_{k_0}^{\infty}\frac{\hbar\tilde{\omega}(k)k^2\,dk}{\exp(\beta\hbar\tilde{\omega}(k))-1}\,, \tag{4.5}$$

where $\tilde{\omega}=\omega_{\mathrm{R}}\exp(-2\omega_{\mathrm{I}}t)$, cf. (3.15), and $k_0:=m_{\mathrm{t}}c/\hbar$ is the smallest possible value of $k:=|\mathbf{k}|$, attained for $\omega=0$ according to the dispersion relation (3.7). ω_{I} depends on k via $\omega_{\mathrm{R}}(k)$, cf. (3.8). We will frequently write ω for ω_{R}, and replace in (3.7) m_{t} by $m_{\mathrm{t}}c/\hbar$, see after (2.3). To account for two transversal degrees, we have to multiply $\log Z$ as well as U by a factor of two, so that the spectral and internal energy densities of transversal tachyonic gray-body radiation read

$$\rho_{\mathrm{T}}(\omega) := \frac{1}{\pi^2}\frac{\hbar\omega k^2(\omega)k'(\omega)}{\exp(\tilde{\beta}\hbar\omega)-1}, \quad \tilde{\beta}(k,t) := \beta \mathrm{e}^{-2\omega_{\mathrm{I}}t}, \quad \frac{U_{\mathrm{T}}}{V} = \int_0^\infty \rho_{\mathrm{T}}(\omega)\,\mathrm{d}\omega\,. \tag{4.6}$$

$k(\omega)$ is defined in (3.7), and $k'(\omega) = \mathrm{d}k/\mathrm{d}\omega$ coincides with (3.13). We have here expanded (i.e., dropped) the damping factor $\exp(-2\omega_{\mathrm{I}}(k)t)$ in the nominator of the integrand in (4.5), and in the denominator we have scaled this factor into the temperature variable, appealing to small ω_{I} and adiabatic variation on time scales $t \ll 1/(2\omega_{\mathrm{I}})$, otherwise the use of an equilibrium distribution would not be justified. As pointed out in (3.22), one may assume $\omega_{\mathrm{I}}(k)$ small in the whole frequency range. The cosmic tachyon background also requires a conformal time scaling of the temperature with the cosmic expansion factor, $\beta = a(t)/(kT)$.

We derive some limit cases of the density (4.6), with permeabilities as in (3.19). $k(\omega)$ and $k'(\omega)$ are explicit in (3.7) and (3.13). In the high frequency limit, we find $k \sim \omega/c$. For $\omega \to 0$, we may approximate $k \sim m_{\mathrm{t}}$ and $k' \sim \omega m_{\mathrm{t}}^{-1}c^{-2}(1+\alpha_0/\omega_0^2)$, see after (3.21). Hence, for high frequencies and with permeabilities as suggested in (3.19)–(3.22), $\rho_{\mathrm{T}}(\omega)$ converges to the Wien limit of the photonic Planck distribution, and in the Rayleigh–Jeans limit we find a linear frequency scaling,

$$\rho_{\mathrm{T}}(\omega\to\infty) \sim \frac{\hbar\omega^3}{\pi^2c^3}\,\mathrm{e}^{-\tilde{\beta}_\infty\hbar\omega}, \quad \rho_{\mathrm{T}}(\omega\to 0) \sim \frac{m_{\mathrm{t}}}{\pi^2 c\hbar\tilde{\beta}_0}(1+\alpha_0/\omega_0^2)\omega\,. \tag{4.7}$$

Here we defined $\tilde{\beta}_{\infty,0} := \beta\mathrm{e}^{-2\omega_{\mathrm{I}}t}$, with the limits $\omega_{\mathrm{I}}(\omega\to\infty,0)$ substituted, cf. (3.22). In the limit $\hat{\varepsilon} = \hat{\mu} = 1$, that is $\alpha_0 \to 0$ in (3.22), we recover from (4.6) the spectral energy density of tachyonic black-body radiation [14,18],

$$\rho_{\mathrm{T}}(\omega) = \frac{\hbar}{\pi^2c^3}\,\frac{\omega^2\sqrt{\omega^2+(m_{\mathrm{t}}c^2/\hbar)^2}}{\exp(\beta\hbar\omega)-1}\,. \tag{4.8}$$

We turn to the coupling of the tachyon field to matter, and study a subluminal, non-relativistic and spinless quantum particle carrying tachyonic charge q. We start with the Schrödinger equation for an attractive Coulomb potential V, coupled by minimal substitution to the tachyon field,

$$\frac{\hbar}{i}\,\partial_t^A\psi = \left(\frac{\hbar^2}{2m}\nabla^A\nabla^A - V - E_0\right)\psi\,, \quad V = -\frac{Ze^2}{4\pi}\frac{1}{r}, \quad E_0 = \frac{m}{2\hbar^2}\frac{Z^2e^4}{(4\pi)^2}\,,$$

$$\partial_t^A := \partial_t - \mathrm{i}q/(\hbar c)A_0, \quad \nabla^A := \nabla - \mathrm{i}q/(\hbar c)\mathbf{A}\,, \tag{4.9}$$

so that the tachyonic charge density in (2.1) reads $\rho = q\psi^*\psi$. If $A_\mu = 0$, we find the discrete hydrogen-like spectrum $E_n = (1-1/n^2)E_0$. The ionization energy E_0 has been inserted in (4.9) to define a zero ground state energy E_1. In this section we consider transversal fields $\mathbf{A}^{\mathrm{T}}$, so that $A_0 = 0$, cf. (2.13). The interaction Hamiltonian can be readily extracted from (4.9),

$$H_{\mathrm{int}}^{\mathrm{T}} = \frac{\mathrm{i}q}{c}\frac{\hbar}{m}\int \psi^*\mathbf{A}^{\mathrm{T}}\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\,, \tag{4.10}$$

where we dropped terms quadratic in $\mathbf{A}^{\mathrm{T}}$. The bound states of the unperturbed Coulomb problem are expanded as $\psi = \sum_n b_n u_n \mathrm{e}^{-\mathrm{i}E_n t/\hbar}$, with time-separated normalized eigenfunctions, $\int u_m u_n^* \,\mathrm{d}\mathbf{x} = \delta_{nm}$, resulting in a bound state energy functional $H(\psi) = \sum_n E_n b_n^* b_n$. As above, we substitute for the Fourier amplitudes b_n statistical operators, writing b_n^+ for the adjoint (instead of b_n^*). These operators may satisfy Bose statistics as above, or, equally well, the anticommutation relations of Fermi statistics, $[b_i, b_j^+]_+ = \delta_{ij}$, $[b_i, b_j]_+ = 0$ and $[b_i^+, b_j^+]_+ = 0$. We will consider a single subluminal particle, and so the statistics does not matter. The particle number operators $N_i := b_i^+ b_i$ are in either case Hermitian and commute. As for the fermionic occupation number representation, the n_i are restricted to 0 and 1, and the Fermi operators are defined by

$$b_i|n_1, \ldots, n_i, \ldots, n_\infty\rangle = (-)^{n_{<i}} n_i |n_1, \ldots, 1 - n_i, \ldots, n_\infty\rangle ,$$

$$b_i^+|n_1, \ldots, n_i, \ldots, n_\infty\rangle = (-)^{n_{<i}} (1 - n_i)|n_1, \ldots, 1 - n_i, \ldots, n_\infty\rangle , \tag{4.11}$$

with $n_{<i} := \sum_{k=1}^{i-1} n_k$, so that the anticommutation relations are satisfied and $N_i|n\rangle = n_i|n\rangle$. It is assumed that the statistical operators a_k of the tachyon field and their adjoints commute with the $b_i^{(+)}$. The interaction Hamiltonian (4.10) can be represented in the tensorial product of the vector spaces used in (4.1) and (4.11). The statistical operators are extended to the product space by $b_n^{(+)} \otimes \mathrm{id}$ and $\mathrm{id} \otimes a_n^{(+)}$, so that these two sets commute.

We study induced tachyon radiation and spontaneous emission of tachyons, based on the interaction (4.10). The initial and final states for absorption read

$$|i\rangle_{\mathrm{abs}} = |0, \ldots, 1_i, \ldots, 0_j, \ldots\rangle \otimes |n_1, \ldots, n_k, \ldots, n_\infty\rangle ,$$

$$|f\rangle_{\mathrm{abs}} = |0, \ldots, 0_i, \ldots, 1_j, \ldots\rangle \otimes |n_1, \ldots, n_k - 1, \ldots, n_\infty\rangle . \tag{4.12}$$

The first factor represents a single subluminal particle, and the second a set of tachyons distributed over some energy range. As for emission,

$$|i\rangle_{\mathrm{em}} = |0, \ldots, 0_i, \ldots, 1_j, \ldots\rangle \otimes |n_1, \ldots, n_k, \ldots, n_\infty\rangle ,$$

$$|f\rangle_{\mathrm{em}} = |0, \ldots, 1_i, \ldots, 0_j, \ldots\rangle \otimes |n_1, \ldots, n_k + 1, \ldots, n_\infty\rangle . \tag{4.13}$$

The first factor in (4.12) and (4.13) is a basis vector for Bose or Fermi statistics. In the second factor (bosonic), $k = (\mathbf{k}, \lambda)$ labels the tachyonic occupation number diminished or augmented when passing from the initial to the final state. As at the beginning of this section, we drop the index λ and consider a linearly polarized tachyon field, so that we can identify k with $\mathbf{k}$ in the subsequent summations.

We substitute the tachyonic and subluminal statistical operators into the interaction Hamiltonian (4.10). The transversal tachyonic wave operator of linear polarization λ is defined by (2.6) and (3.14),

$$\mathbf{A}^{\mathrm{T}} = L^{-3/2} \sum_k (\alpha_k^{\mathrm{T}} a_k \boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \mathrm{e}^{\mathrm{i}(\mathbf{kx} - \omega_k^* t)} + \mathrm{h.c.}) , \tag{4.14}$$

with $a_k^{(+)}$ as in (4.1). The bound state wave operator ψ is defined after (4.10). Using the hermiticity of $i\nabla$ in (4.10), we find the interaction operator as

$$H_{\rm int}^{\rm T} = L^{-3/2}\,\frac{{\rm i}q\hbar}{mc}\sum_{i,j,k} b_j^{+} b_i \alpha_k^{\rm T} a_k \int u_j^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_i {\rm e}^{{\rm i}\mathbf{k}x}\,{\rm d}\mathbf{x} e^{{\rm i}(\omega_j-\omega_i-\omega_k^*)t}$$

$$+ L^{-3/2}\,\frac{{\rm i}q\hbar}{mc}\sum_{i,j,k} b_j^{+} b_i \alpha_k^{\rm T} a_k^{+} \int u_j^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_i {\rm e}^{-{\rm i}\mathbf{k}x}\,{\rm d}\mathbf{x} e^{{\rm i}(\omega_j-\omega_i+\omega_k)t}\,. \tag{4.15}$$

The summation index k stands for $\mathbf{k}$, and the polarization only enters via $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$. [In (4.12) and (4.13), the indices i, j and k are fixed and should not be confused with summation indices.] We so find, cf. (4.1), (4.11)–(4.13) and (4.15),

$$\langle f|H_{\rm int}^{\rm T}|i\rangle_{\rm abs} = \langle T_{\rm abs}^{\rm T}\rangle {\rm e}^{{\rm i}(\omega_j-\omega_i-\omega_k^*)t},$$

$$\langle T_{\rm abs}^{\rm T}\rangle := \frac{{\rm i}q\hbar}{mc}\,\frac{\alpha_k^{\rm T}}{L^{3/2}}\sqrt{n_k}\int u_j^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_i {\rm e}^{{\rm i}\mathbf{k}x}\,{\rm d}\mathbf{x}\,, \tag{4.16}$$

$$\langle f|H_{\rm int}^{\rm T}|i\rangle_{\rm em} = \langle T_{\rm em}^{\rm T}\rangle {\rm e}^{-{\rm i}(\omega_j-\omega_i-\omega_k)t}\,,$$

$$\langle T_{\rm em}^{\rm T}\rangle := \frac{{\rm i}q\hbar}{mc}\,\frac{\alpha_k^{\rm T}}{L^{3/2}}\sqrt{n_k+1}\int u_i^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_j {\rm e}^{-{\rm i}\mathbf{k}x}\,{\rm d}\mathbf{x}\,. \tag{4.17}$$

These matrix elements are independent of the statistics used in defining $b_i^{(+)}$, since the subluminal factors in (4.12) and (4.13) are one-particle states. n_k is a tachyonic occupation number in Bose statistics.

The transition rate for induced absorption, with a given linear polarization λ, is obtained by a standard procedure,

$$w_{\rm abs}^{\rm ind} \sim \frac{1}{t\hbar^2}\sum_k |\langle T_{\rm abs}\rangle|^2 \left|\int_0^t {\rm e}^{{\rm i}(\omega_{\rm E}+{\rm i}\omega_{\rm I})t}\,{\rm d}t\right|^2, \tag{4.18}$$

$$\omega_{\rm E} := \omega_j - \omega_i - \omega_{\rm R}, \quad \omega_k =: \omega_{\rm R}(k) + {\rm i}\omega_{\rm I}(k)\,. \tag{4.19}$$

(The same formula also applies to transitions effected by longitudinal fermionic tachyons, cf. Section 5, so we do not indicate here the T-superscripts in $w_{\rm abs}^{\rm T,ind}$ and $T_{\rm abs}^{\rm T}$.) For arbitrary real numbers $\omega_{\rm E,I}$,

$$\frac{1}{t}\left|\int_0^t {\rm e}^{{\rm i}(\omega_{\rm E}+{\rm i}\omega_{\rm I})t}\,{\rm d}t\right|^2 = \frac{4}{t}\,{\rm e}^{-\omega_{\rm I}t}\,\frac{\sin^2(\omega_{\rm E}t/2)+\sinh^2(\omega_{\rm I}t/2)}{\omega_{\rm E}^2+\omega_{\rm I}^2} =: F(\omega_{\rm E},\omega_{\rm I})\,, \tag{4.20}$$

$$\int_{-\infty}^{+\infty} F(\omega_{\rm E},\omega_{\rm I})\,{\rm d}\omega_{\rm R} = 2\pi{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)/(\omega_{\rm I}t)\,, \tag{4.21}$$

and we assume $\omega_{\rm I} \geqslant 0$, $t \gg 1$, $\omega_{\rm I}t \ll 1$, so that $F(\omega_{\rm E},\omega_{\rm I})$ is strongly peaked as a function of $\omega_{\rm E}$, with maximum $F(\omega_{\rm E}\approx 0,\omega_{\rm I}) \approx t$. Due to dissipation, cf. Section 3, energy conservation is only approximate, ${\rm Re}\,\omega_k \approx \omega_j - \omega_i$, hence $\omega_{\rm E} \approx 0$.

Like in (4.4) and (4.5), we replace the box-summation in (4.18) by the continuum limit

$$\sum_k n_k \to \frac{L^3}{(2\pi)^3} \int \frac{\mathrm{d}\mathbf{k}}{\mathrm{e}^{\beta\hbar\tilde{\omega}_k} - 1}, \quad \mathrm{d}\mathbf{k} = k^2 k'(\omega_\mathrm{R})\, \mathrm{d}\omega_\mathrm{R}\, \mathrm{d}\Omega\,, \tag{4.22}$$

with $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ as solid angle element. Explicit formulas for $k(\omega_\mathrm{R})$ and its derivative are given in (3.7) and (3.13). The ω_R-integration in (4.18) is then carried out by steepest descent, at $\omega_{ji} := \omega_j - \omega_i$,

$$w_\mathrm{abs}^\mathrm{ind} \sim \frac{1}{\pi} \frac{q^2}{4\pi} \frac{1}{\hbar^2 c^2} \frac{k^2 k' \alpha_k^{\mathrm{T}^2}}{\mathrm{e}^{\beta\hbar\tilde{\omega}} - 1} \frac{\mathrm{e}^{-\omega_\mathrm{I} t} \sinh(\omega_\mathrm{I} t)}{\omega_\mathrm{I} t} \left| \frac{\hbar}{m} \int u_j^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \cdot \nabla u_i \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}\, \mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\Omega$$

$$=: \tfrac{1}{2} B_{ij}^\mathrm{T}(-\mathbf{k}, \lambda) \rho_\mathrm{T}(\omega_{ji})\, \mathrm{d}\Omega\,. \tag{4.23}$$

ω_I is by virtue of (3.8) a function of ω_R, positive and uniformly bounded, cf. the end of Section 3, so that the restrictions mentioned after (4.21) are satisfied over the whole integration range. The steepest descent procedure boils down to (4.21); ω_I is taken at $\omega_\mathrm{R} = \omega_{ji}$, likewise the remaining ω_R-dependent factors in (4.18), and $\tilde{\omega} = \omega_{ji} \exp(-2\omega_\mathrm{I}(\omega_{ji})t)$, cf. (3.15). The direction of the tachyonic wave vector $\mathbf{k}$ is specified by the angular variables in $\mathrm{d}\Omega$, and its magnitude by $k(\omega_{ji})$, cf. (3.7). The coefficient α_k^T in (3.14) is likewise taken at $\omega_\mathrm{R} = \omega_{ji}$. The spectral density ρ_T is defined in (4.6). In short, $w_\mathrm{abs}^\mathrm{T,ind}$ in (4.23) is the transition probability per unit time for an electron to be moved from an initial state ω_i to an excited state ω_j by the absorption of a transversal tachyon $(\mathbf{k}, \lambda)$.

The transition rate for emission is

$$w_\mathrm{em} \sim \frac{1}{t\hbar^2} \sum_k |\langle T_\mathrm{em} \rangle|^2 \left| \int_0^t \mathrm{e}^{-\mathrm{i}(\omega_\mathrm{E} - \mathrm{i}\omega_\mathrm{I})t}\, \mathrm{d}t \right|^2\,, \tag{4.24}$$

with $\omega_\mathrm{E,I}$ defined in (4.19). (ω_j is now the excited initial state and ω_i the final state.) Like in (4.18), we have dropped here the T-superscripts of w_em^T and T_em^T, cf. (4.17). The time integration gives the same result as for absorption, cf. (4.20), and thus the ω_R-integration in the continuum limit (4.22) is the same as in (4.21). The crucial difference to absorption is the $n_k + 1$ factor in $|\langle T_\mathrm{em}^\mathrm{T} \rangle|^2$, contrary to the n_k-proportionality of $|\langle T_\mathrm{abs}^\mathrm{T} \rangle|^2$, cf. (4.17) and (4.16). We split $w_\mathrm{em}^\mathrm{T} := w_\mathrm{em}^\mathrm{T,ind} + w_\mathrm{em}^\mathrm{T,sp}$, where $w_\mathrm{em}^\mathrm{T,ind}$ denotes the contribution of the n_k-proportional terms in (4.24), responsible for induced emission. A procedure completely analogous to the foregoing gives

$$w_\mathrm{em}^\mathrm{T,ind} \sim \frac{1}{\pi} \frac{q^2}{4\pi} \frac{1}{\hbar^2 c^2} \frac{k^2 k' \alpha_k^{\mathrm{T}^2}}{\mathrm{e}^{\beta\hbar\tilde{\omega}} - 1} \frac{\mathrm{e}^{-\omega_\mathrm{I} t} \sinh(\omega_\mathrm{I} t)}{\omega_\mathrm{I} t} \left| \frac{\hbar}{m} \int u_i^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \cdot \nabla u_j \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}}\, \mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\Omega$$

$$=: \tfrac{1}{2} B_{ji}^\mathrm{T}(\mathbf{k}, \lambda) \rho_\mathrm{T}(\omega_{ji})\, \mathrm{d}\Omega\,, \tag{4.25}$$

where all factors outside the integral are taken at $\omega_\mathrm{R} = \omega_{ji} = \omega_j - \omega_i$ as in (4.23), and $\alpha_k^{\mathrm{T}^2}$ relates to $\mathrm{d}\omega_\mathrm{R}/\mathrm{d}k$ as indicated after (3.14). Applying Green's formula, we recover the symmetry of Einstein's B-coefficients, $B_{ji}^\mathrm{T}(\mathbf{k}, \lambda) = B_{ij}^\mathrm{T}(-\mathbf{k}, \lambda)$. The transversal spontaneous emission rate $w_\mathrm{em}^\mathrm{T,sp}$ is obtained by dropping the $n_k + 1$-factor of $|\langle T_\mathrm{em}^\mathrm{T} \rangle|^2$

in (4.24). The k-summation in (4.24) is replaced by the integration indicated after (4.4), and we find, via (4.20) and (4.21),

$$w_{\rm em}^{\rm T,sp} \sim ({\rm e}^{\beta\hbar\tilde{\omega}} - 1) w_{\rm em}^{\rm T,ind} =: A_{ji}^{\rm T}(\mathbf{k},\lambda)\,{\rm d}\Omega\,. \tag{4.26}$$

The A- and B-coefficients for transversal tachyon radiation thus read

$$A_{ji}^{\rm T}(\mathbf{k},\lambda) = \hbar\omega_{ji}k^2k' B_{ji}^{\rm T}(\mathbf{k},\lambda)/(2\pi^2)$$
$$= \frac{q^2}{4\pi}\frac{k}{\hbar c^2}\frac{{\rm Re}\,\hat{\mu}}{2\pi}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t}\left|\frac{\hbar}{m}\int u_i^* \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_j {\rm e}^{-{\rm i}\mathbf{k}\mathbf{x}}\,{\rm d}\mathbf{x}\right|^2 . \tag{4.27}$$

A consistency check of (4.27) and (4.6) is provided by the equilibrium balancing of emission and absorption events,

$$N_j(\tfrac{1}{2}B_{ji}^{\rm T}(\mathbf{k},\lambda)\rho_{\rm T}(\omega_{ji}) + A_{ji}^{\rm T}(\mathbf{k},\lambda)) = \tfrac{1}{2}N_i B_{ij}^{\rm T}(-\mathbf{k},\lambda)\rho_{\rm T}(\omega_{ji}) \tag{4.28}$$

with occupation numbers related by their weight factors as $N_j/N_i = \exp(-\tilde{\beta}\hbar\omega_{ji})$, where we used $\beta\tilde{\omega} = \tilde{\beta}\omega_{ji}$ and $\tilde{\beta} = \beta\exp(-2\omega_{\rm I}(\omega_{ji})t)$, cf. (4.6). This can also be regarded as a derivation of the spectral density $\rho_{\rm T}(\omega)$ alternative to (4.2)–(4.6), when combined with (4.27). The factors of one-half in (4.23), (4.25) and (4.28) just indicate that we consider a single transversal degree, that is, modes of a fixed polarization, to compare better to the longitudinal radiation discussed in Section 5. The transition rates for unpolarized radiation are obtained by replacing $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla$ in (4.23), (4.25) and (4.27) by the transversal component of the gradient, $\nabla^{\rm T} := \nabla - \mathbf{k}_0(\mathbf{k}_0\cdot\nabla)$.

In dipole approximation, we may drop the exponential ${\rm e}^{\pm{\rm i}\mathbf{k}\mathbf{x}}$ in the preceding integrals and use the identity

$$\frac{\hbar}{m}\int u_i^*\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_j\,{\rm d}\mathbf{x} = \omega_{ji}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\mathbf{d}_{ji},\quad \mathbf{d}_{ji} := \int u_i^*\mathbf{r}u_j\,{\rm d}\mathbf{x}\,, \tag{4.29}$$

which is also valid for longitudinal polarization, i.e., for $\boldsymbol{\varepsilon}_{\mathbf{k},3}$, as well as for unpolarized transversal radiation via the substitutions $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla \to \nabla^{\rm T}$ and $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\mathbf{d}_{ji} \to \mathbf{d}_{ji} - \mathbf{k}_0(\mathbf{k}_0\cdot\mathbf{d}_{ji})$. As for the latter, the angular integration in (4.23), (4.25) and (4.26) can easily be carried out with $\mathbf{d}_{ji}$ as polar axis and $\int\sin^2\theta\,{\rm d}\Omega = 8\pi/3$. We so find the unpolarized total transversal transition rates as

$$w_{\rm abs,em}^{\rm T,ind(d)} \sim \frac{4}{3}\frac{q^2}{4\pi}\frac{k}{\hbar}\frac{\omega_{ji}^2}{c^2}\frac{{\rm Re}\,\hat{\mu}}{{\rm e}^{\tilde{\beta}\hbar\omega_{ji}} - 1}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t}|\mathbf{d}_{ji}|^2 =: B_{ji}^{\rm (T,d)}\rho_{\rm T}(\omega_{ji})\,, \tag{4.30}$$

$$w_{\rm em}^{\rm T,sp(d)} \sim ({\rm e}^{\tilde{\beta}\hbar\omega_{ji}} - 1)w_{\rm em}^{\rm T,ind(d)} =: A_{ji}^{\rm (T,d)}\,. \tag{4.31}$$

The detailed balancing condition (4.28) apparently also applies to $A^{\rm (T,d)}$ and $B^{\rm (T,d)}$, the Einstein coefficients in dipole approximation for unpolarized transversal radiation in all directions.

Finally we shortly discuss the tachyonic analog to the photoelectric effect, the transfer of a bound electron into the continuum by the absorption of a tachyon. The Coulomb wave functions of the continuum are approximated by plane waves, that is, we put $u_j = L^{-3/2}{\rm e}^{{\rm i}\mathbf{k}_{\rm e}\mathbf{x}}$ in the matrix element (4.16) (Born approximation) to maintain the discrete states assumed in (4.1) and (4.11), and we write u_0 for the initial bound state (not necessarily the ground state). In formula (4.18) for the absorptive transition rate $w_{\rm abs}^{\rm ind}$, we take the k-summation over the electronic (rather than tachyonic) wave vectors. (There is still only one electron, so that the statistics used is irrelevant.) In the continuum limit, we may replace this summation by

$$L^3/(2\pi)^3 \int \mathrm{d}\mathbf{k}_{\rm e}, \quad \mathrm{d}\mathbf{k}_{\rm e} = (m/\hbar)k_{\rm e}\,\mathrm{d}\omega_{\rm e}\,\mathrm{d}\Omega_{\rm e}, \quad \omega_{\rm e} = \hbar k_{\rm e}^2/(2m)\,; \tag{4.32}$$

the solid angle is now defined with the angular variables of the electronic wave vector. We also write, instead of (4.19), $\omega_{\rm E} := \omega_{\rm e} - \omega_0 - \omega_{\rm R}(k)$. Here, $\omega_{0,\rm e}$ are the frequencies of u_0 and u_j, respectively, and $\mathbf{k}$ and $\omega_k = \omega_{\rm R} + \mathrm{i}\omega_{\rm I}$ refer to the tachyons. The $\mathrm{d}\omega_{\rm e}$-integration is carried out by steepest descent around $\omega_{\rm e} = \omega_0 + \omega_{\rm R}$, by making use of (4.21). Energy conservation, $\omega_{\rm E} \approx 0$, is again only approximate, unless the dissipation generating $\omega_{\rm I}$ vanishes. We so find

$$w_{\rm abs}^{\rm T,ind} \sim \frac{1}{\pi}\frac{n_k}{L^3}\frac{q^2}{4\pi}\frac{k_{\rm e}\alpha_k^{\rm T^2}}{mc^2\hbar}\frac{\mathrm{e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t}\left|\int \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\nabla u_0\,\mathrm{d}\mathbf{x}\right|^2 \mathrm{d}\Omega_{\rm e}\,. \tag{4.33}$$

The integral can be replaced by $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\mathbf{k}_{\rm e}\int \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}u_0\,\mathrm{d}\mathbf{x}$, the electronic $\mathbf{k}_{\rm e}$ is taken at $\omega_{\rm e} \approx \omega_{\rm R}$, cf. (4.32), as $\omega_{\rm R} \gg \omega_0$ is required by the Born approximation. The tachyonic $k(\omega_{\rm R})$ and $\omega_{\rm I}(\omega_{\rm R})$ are defined in (3.7) and (3.8), and the amplitude $\alpha_k^{\rm T^2}$ in (3.14). Dividing $w_{\rm abs}^{\rm T,ind}$ by the incoming tachyonic flux density, $|\mathbf{v}_{\rm gr}|n_k/L^3$, cf. (3.13), we find the differential cross section for polarized transversal radiation,

$$\mathrm{d}\sigma^{\rm T} \sim \frac{1}{2\pi}\frac{q^2}{4\pi}\frac{\mathrm{Re}\,\hat{\mu}}{mc^2}\frac{k_{\rm e}}{k}\frac{\mathrm{e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t}|\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\mathbf{k}_{\rm e}|^2\left|\int \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}u_0\,\mathrm{d}\mathbf{x}\right|^2 \mathrm{d}\Omega_{\rm e}\,. \tag{4.34}$$

The unpolarized transversal section is obtained by replacing $|\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\cdot\mathbf{k}_{\rm e}|^2$ by $\frac{1}{2}(|\mathbf{k}_{\rm e}|^2 - (\mathbf{k}_{\rm e}\cdot\mathbf{k}_0)^2)$, or by $\frac{1}{2}|\mathbf{k}_{\rm e}|^2\sin^2\theta$ with $\mathbf{k}$ as polar axis. A phenomenological discussion of (4.34), with regard to the tachyonic ionization of Rydberg atoms, is given in Ref. [14], where (4.34) was derived semiclassically in the limit $\omega_{\rm I} = 0$, neglecting dissipation and the resulting adiabatic damping.

5. Longitudinal tachyons in Fermi–Dirac statistics

To render the longitudinal energy density in (3.16) positive, we quantize the modes in Fermi statistics, cf. (4.11), otherwise the reasoning is the same as in Section 4. Like in (4.2), we replace in the longitudinal density the Fourier coefficients by statistical operators, and make use of the anticommutation relation $[a_i, a_j^+]_+ = \delta_{ij}$, which effects the sign change turning the energy density into a positive definite operator,

$$\left\langle \int \rho_{\rm E}^{\rm L} \right\rangle = \hbar\sum_{\mathbf{k}}\tilde{\omega}a^+(\mathbf{k},3)a(\mathbf{k},3), \quad \left\langle n\left|\int \rho_{\rm E}^{\rm L}\right|n\right\rangle = E(n_1,\ldots,n_\infty) = \sum_{i=1}^{\infty}\hbar\tilde{\omega}_i n_i\,. \tag{5.1}$$

The $a^{(+)}$ admit the representation (4.11), so that the n_i are restricted to zero and one. Partition function and internal energy read, cf. (4.3) and (4.4),

$$Z = \sum_{n_1,\ldots,n_\infty=0,1} \exp(-\beta E(n_1,\ldots,n_\infty)) = \prod_{\mathbf{k}} (1+\exp(-\beta\hbar\tilde{\omega}(|\mathbf{k}|,t)))\,, \tag{5.2}$$

$$\log Z = \sum_{\mathbf{k}} \log(1+\exp(-\beta\hbar\tilde{\omega})), \quad U = \sum_{\mathbf{k}} \frac{\hbar\tilde{\omega}}{\exp(\beta\hbar\tilde{\omega})+1}\,. \tag{5.3}$$

In the thermodynamic limit, we find the spectral energy density and the internal energy of the longitudinal modes as, cf. (4.5) and (4.6),

$$\rho_{\mathrm{L}}(\omega) := \frac{1}{2\pi^2}\frac{\hbar\omega k^2(\omega)k'(\omega)}{\exp(\tilde{\beta}\hbar\omega)+1}, \quad U_{\mathrm{L}}/V = \int_0^\infty \rho_{\mathrm{L}}(\omega)\,\mathrm{d}\omega\,. \tag{5.4}$$

Apparently, ρ_{L} differs from the transversal density ρ_{T} only by a sign change in the denominator and a factor of 1/2. (There is only one longitudinal degree of freedom.) As for the number density of the longitudinal modes, this is of course $n_{\mathrm{L}}(\omega) := \rho_{\mathrm{L}}(\omega)/(\hbar\omega)$, so that $N_{\mathrm{L}}/V = \int_0^\infty n_{\mathrm{L}}(\omega)\,\mathrm{d}\omega$, which will be made more explicit in the black-body limit at the end of this section. In the high frequency regime, we find $\rho_{\mathrm{L}} \sim \rho_{\mathrm{T}}/2$, and the Rayleigh–Jeans limit is quadratic in the frequency and temperature independent, $\rho_{\mathrm{L}} \sim (\hbar\omega\tilde{\beta}_0/4)\rho_{\mathrm{T}}$. (The respective limits (4.7) of the transversal density are to be inserted.) In the black-body limit ($\hat{\varepsilon} = \hat{\mu} = 1,\ \omega_{\mathrm{I}} = 0$),

$$\rho_{\mathrm{L}}(\omega) = \frac{\hbar}{2\pi^2 c^3}\frac{\omega^2\sqrt{\omega^2+(m_{\mathrm{t}}c^2/\hbar)^2}}{\exp(\beta\hbar\omega)+1}\,. \tag{5.5}$$

Next, we calculate the Einstein coefficients for longitudinal radiation. The interaction Hamiltonian is readily found, cf. (4.9) and (4.10),

$$H^{\mathrm{L}}_{\mathrm{int}} = \frac{iq}{c}\frac{\hbar}{m}\int \psi^*\mathbf{A}^{\mathrm{L}}\cdot\nabla\psi\,\mathrm{d}\mathbf{x} - \frac{q}{c}A_0\int \psi^*\psi\,\mathrm{d}\mathbf{x}\,. \tag{5.6}$$

The spatial component of the longitudinal wave operator reads, cf. (2.6) and (3.14),

$$\mathbf{A}^{\mathrm{L}} = L^{-3/2}\sum_k (\alpha^{\mathrm{L}}_k a_k \boldsymbol{\varepsilon}_{\mathbf{k},3}\mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega^*_k t)} + \mathrm{h.c.})\,, \tag{5.7}$$

and the time component follows from (2.13) via $\hat{A}_0(\mathbf{k}) = \alpha^{\mathrm{L}}_0\alpha^{\mathrm{L}}a(\mathbf{k},3)$ and the expansion (3.6)–(3.8),

$$A_0 = L^{-3/2}\sum_k (\alpha^{\mathrm{L}}_{0k}\alpha^{\mathrm{L}}_k a_k \mathrm{e}^{\mathrm{i}(\mathbf{kx}-\omega^*_k t)} + \mathrm{h.c.}), \quad \alpha^{\mathrm{L}}_0 := \frac{-c^2 k\omega^*}{\omega_{\mathrm{R}}^2\,\mathrm{Re}\,\hat{\varepsilon}\,\mathrm{Re}\,\hat{\mu}} + \mathrm{O}(\omega_{\mathrm{I}}^2)\,. \tag{5.8}$$

We also restore $m_{\mathrm{t}} \to m_{\mathrm{t}}c/\hbar$ in k and α^{L}, cf. (3.7) and (3.14). Hence we may write the interaction as

$$H^{\mathrm{L}}_{\mathrm{int}} = L^{-3/2}\frac{\mathrm{i}q\hbar}{mc}\sum_{i,j,k} b^+_j b_i \alpha^{\mathrm{L}}_k a_k \int u^*_j \boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_i \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}\,\mathrm{d}\mathbf{x}\mathrm{e}^{\mathrm{i}(\omega_j-\omega_i-\omega^*_k)t}$$
$$- L^{-3/2}\frac{q}{c}\sum_{i,j,k} b^+_j b_i \alpha^{\mathrm{L}}_{0k}\alpha^{\mathrm{L}}_k a_k \int u^*_j u_i \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}\,\mathrm{d}\mathbf{x}\mathrm{e}^{\mathrm{i}(\omega_j-\omega_i-\omega^*_k)t}$$

$$+L^{-3/2}\frac{\mathrm{i}q\hbar}{mc}\sum_{i,j,k}b_j^+b_i\alpha_k^{\mathrm{L}}a_k^+\int u_j^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_i\mathrm{e}^{-\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}e^{\mathrm{i}(\omega_j-\omega_i+\omega_k)t}$$

$$-L^{-3/2}\frac{q}{c}\sum_{i,j,k}b_j^+b_i\alpha_{0k}^{\mathrm{L}*}\alpha_k^{\mathrm{L}}a_k^+\int u_j^*u_i\mathrm{e}^{-\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}e^{\mathrm{i}(\omega_j-\omega_i+\omega_k)t}\ . \tag{5.9}$$

The operators $a^{(+)}$ are assumed in the representation (4.11). The $b^{(+)}$ of the subluminal particle may satisfy Bose or Fermi statistics, and we use for them the representations (4.1) or (4.11). The $b^{(+)}$ and $a^{(+)}$ commute. Thus we may take the initial and final states defined in (4.12) and (4.13), but the particle numbers n_i in the second tensorial factor are now restricted to zero and one.

As for the absorption of a longitudinal fermionic tachyon, with initial and final states defined in (4.12), we find

$$\langle f|H_{\mathrm{int}}^{\mathrm{L}}|i\rangle_{\mathrm{abs}}=\langle T_{\mathrm{abs}}^{\mathrm{L}}\rangle\mathrm{e}^{\mathrm{i}(\omega_j-\omega_i-\omega_k^*)t}\ ,$$

$$\langle T_{\mathrm{abs}}^{\mathrm{L}}\rangle:=\frac{q}{c}\frac{\alpha_k^{\mathrm{L}}}{L^{3/2}}(-)^{n_{<k}}n_k\left(\frac{\mathrm{i}\hbar}{m}\int u_j^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_i\mathrm{e}^{\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}-\alpha_{0k}^{\mathrm{L}}\int u_j^*u_i\mathrm{e}^{\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}\right)\ , \tag{5.10}$$

and the matrix element for emission is likewise easily assembled from (5.9), (4.13) and (4.11),

$$\langle f|H_{\mathrm{int}}^{\mathrm{L}}|i\rangle_{\mathrm{em}}=\langle T_{\mathrm{em}}^{\mathrm{L}}\rangle\mathrm{e}^{-\mathrm{i}(\omega_j-\omega_i-\omega_k)t}\ ,$$

$$\langle T_{\mathrm{em}}^{\mathrm{L}}\rangle:=\frac{q}{c}\frac{\alpha_k^{\mathrm{L}}}{L^{3/2}}(-)^{n_{<k}}(1-n_k)$$

$$\times\left(\frac{\mathrm{i}\hbar}{m}\int u_i^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_j\mathrm{e}^{-\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}-\alpha_{0k}^{\mathrm{L}*}\int u_i^*u_j\mathrm{e}^{-\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}\right)\ . \tag{5.11}$$

In either case, the n_k only admit the values zero and one, and it is instructive to compare to the transversal bosonic radiation (4.16) and (4.17).

The transition rates can be compiled as in Section 4. Equations (4.18)–(4.21) remain unaltered, as well as (4.22), apart from an obvious sign change in the denominator. We so find the induced absorption rate for longitudinal fermionic quanta, cf. (5.10),

$$w_{\mathrm{abs}}^{\mathrm{L,ind}}=:B_{ij}^{\mathrm{L}}(-\mathbf{k})\rho_{\mathrm{L}}(\omega_{ji})\,\mathrm{d}\Omega\sim\frac{1}{\pi}\frac{q^2}{4\pi}\frac{1}{\hbar^2c^2}\frac{k^2k'\alpha_k^{\mathrm{L}^2}}{\mathrm{e}^{\beta\hbar\tilde{\omega}}+1}\frac{\mathrm{e}^{-\omega_{\mathrm{I}}t}\sinh(\omega_{\mathrm{I}}t)}{\omega_{\mathrm{I}}t}$$

$$\times\left|\frac{\hbar}{m}\int u_j^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_i\mathrm{e}^{\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}-\frac{\mathrm{i}c^2k}{\omega_{\mathrm{R}}\operatorname{Re}\hat{\varepsilon}\operatorname{Re}\hat{\mu}}\int u_j^*u_i\mathrm{e}^{\mathrm{i}\mathbf{k}\boldsymbol{x}}\,\mathrm{d}\mathbf{x}\right|^2\mathrm{d}\Omega\ , \tag{5.12}$$

which is to be compared to (4.23). The factors outside the integrals are again taken at $\omega_{\mathrm{R}}=\omega_{ji}$, and $|\mathbf{k}|=k(\omega_{ji})$ in the exponents. The symmetry $B_{ji}^{\mathrm{L}}(\mathbf{k})=B_{ij}^{\mathrm{L}}(-\mathbf{k})$ follows from the Green formula and the condition $\hbar\omega_{\mathrm{R}}\ll mc^2$, as the subluminal particle is non-relativistic.

Concerning emission, the transition rate is determined by (4.24) with $\langle T_{\mathrm{em}}^{\mathrm{L}}\rangle$ defined in (5.11). To keep the formal analogy to the derivation following (4.24), we write

$w_{\rm em}^{\rm L} = \tilde{w}_{\rm em}^{\rm L,sp} - w_{\rm em}^{\rm L,ind}$, where $-w_{\rm em}^{\rm L,ind}$ stands for the contribution of the n_k-proportional terms in (5.11), and $\tilde{w}_{\rm em}^{\rm L,sp}$ for the remaining terms independent of n_k. The actual spontaneous emission rate $w_{\rm em}^{\rm L,sp}$ will be identified in (5.19). By proceeding as after (4.24), we find

$$w_{\rm em}^{\rm L,ind} =: B_{ji}^{\rm L}(\mathbf{k})\rho_{\rm L}(\omega_{ji})\,{\rm d}\Omega \sim \frac{1}{\pi}\frac{q^2}{4\pi}\frac{1}{\hbar^2 c^2}\frac{k^2 k' \alpha_k^{{\rm L}^2}}{{\rm e}^{\beta\hbar\tilde{\omega}}+1}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t} \times \left|\frac{\hbar}{m}\int u_i^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_j {\rm e}^{-{\rm i}\mathbf{k}\mathbf{x}}\,{\rm d}\mathbf{x} - \frac{{\rm i}c^2 k}{\omega_{\rm R}\,{\rm Re}\,\hat{\varepsilon}\,{\rm Re}\,\hat{\mu}}\int u_i^* u_j {\rm e}^{-{\rm i}\mathbf{k}\mathbf{x}}\,{\rm d}\mathbf{x}\right|^2 {\rm d}\Omega\,, \tag{5.13}$$

$$\tilde{w}_{\rm em}^{\rm L,sp} \sim ({\rm e}^{\beta\hbar\tilde{\omega}}+1)w_{\rm em}^{\rm L,ind} =: \tilde{A}_{ji}^{\rm L}(\mathbf{k})\,{\rm d}\Omega\,. \tag{5.14}$$

We restore the mass unit in $\alpha_k^{\rm L}$, that is, replace $m_{\rm t}$ in (3.14) by $m_{\rm t}c/\hbar$, and so obtain

$$\tilde{A}_{ji}^{\rm L}(\mathbf{k}) = \frac{1}{2\pi^2}\hbar\omega_{ji}k^2 k' B_{ji}^{\rm L}(\mathbf{k}) = \frac{q^2}{4\pi}\frac{k}{\hbar c^2}\frac{{\rm Re}\,\hat{\varepsilon}\,{\rm Re}^2\hat{\mu}}{2\pi}\frac{\hbar^2\omega_{ji}^2}{m_{\rm t}^2 c^4}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t} \times \left|\frac{\hbar}{m}\int u_i^*\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_j {\rm e}^{-{\rm i}\mathbf{k}\mathbf{x}}\,{\rm d}\mathbf{x} - \frac{{\rm i}c^2 k}{\omega_{\rm R}\,{\rm Re}\,\hat{\varepsilon}\,{\rm Re}\,\hat{\mu}}\int u_i^* u_j {\rm e}^{-{\rm i}\mathbf{k}\mathbf{x}}\,{\rm d}\mathbf{x}\right|^2 . \tag{5.15}$$

In this way we recover, via the balancing condition

$$\tilde{A}_{ji}^{\rm L}(\mathbf{k}) - B_{ji}^{\rm L}(\mathbf{k})\rho_{\rm L}(\omega_{ji}) = B_{ij}^{\rm L}(-\mathbf{k})\rho_{\rm L}(\omega_{ji})\exp(\tilde{\beta}\hbar\omega_{ji})\,, \tag{5.16}$$

the equilibrium distribution (5.4). In (5.22), we will write this balance in a more comprehensible form, with the proper A-coefficient.

The dipole approximation of the transition rates (5.12)–(5.14) is obtained by dropping the exponentials in the integrals, so that the second integral in (5.12), (5.13) and (5.15) vanishes, and the first integral is settled by Ehrenfest's theorem (4.29). We so find, in the same notation as in (4.29)–(4.31), the total (that is angular-integrated, $\int\cos^2\theta\,{\rm d}\Omega = 4\pi/3$) longitudinal transition rates

$$w_{\rm abs,em}^{\rm L,ind(d)} \sim \frac{2}{3}\frac{q^2}{4\pi}\frac{k}{\hbar}\frac{\omega_{ji}^2}{c^2}\frac{\hbar^2\omega_{ji}^2}{m_{\rm t}^2 c^4}\frac{{\rm Re}^2\,\hat{\mu}\,{\rm Re}\,\hat{\varepsilon}}{{\rm e}^{\tilde{\beta}\hbar\omega_{ji}}+1}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t}|\mathbf{d}_{ji}|^2 =: B_{ji}^{\rm (L,d)}\rho_{\rm L}(\omega_{ji})\,, \tag{5.17}$$

$$\tilde{w}_{\rm em}^{\rm L,sp(d)} \sim ({\rm e}^{\tilde{\beta}\hbar\omega_{ji}}+1)w_{\rm em}^{\rm L,ind(d)} =: \tilde{A}_{ji}^{\rm (L,d)}\,, \tag{5.18}$$

with $\mathbf{d}_{ji}$ as defined in (4.29). Clearly, $B_{ji}^{\rm (L,d)} = B_{ij}^{\rm (L,d)}$, and the first equality in (5.15) as well as the balancing conditions (5.16) are likewise evident in dipole approximation.

The spontaneous emission rate $w_{\rm em}^{\rm L,sp}$ and the Einstein A-coefficient attached to it are readily identified,

$$w_{\rm em}^{\rm L} = w_{\rm em}^{\rm L,ind} + w_{\rm em}^{\rm L,sp}, \quad w_{\rm em}^{\rm L,sp} := \tilde{w}_{\rm em}^{\rm L,sp} - 2w_{\rm em}^{\rm L,ind}\,, \tag{5.19}$$

$$w_{\rm em}^{\rm L,sp} \sim ({\rm e}^{\tilde{\beta}\hbar\omega_{ji}}-1)w_{\rm em}^{\rm L,ind} =: A_{ji}^{\rm L}(\mathbf{k})\,{\rm d}\Omega\,, \tag{5.20}$$

$$A^{\rm L}_{ji}(\mathbf{k}) = \tanh(\tilde{\beta}\hbar\omega_{ji}/2)\tilde{A}^{\rm L}_{ji}(\mathbf{k})\,, \tag{5.21}$$

$$B^{\rm L}_{ji}(\mathbf{k})\rho_{\rm L}(\omega_{ji}) + A^{\rm L}_{ji}(\mathbf{k}) = B^{\rm L}_{ij}(-\mathbf{k})\rho_{\rm L}(\omega_{ji})\exp(\tilde{\beta}\hbar\omega_{ji})\,. \tag{5.22}$$

It is clear from (5.14) and (5.19) that $w^{\rm L,sp}_{\rm em} > 0$. Relations (5.19)–(5.22) also hold in dipole approximation, and we may compare to the transversal rates, cf. (4.30) and (4.31),

$$\frac{w^{\rm L,sp(d)}_{\rm em}}{w^{\rm T,sp(d)}_{\rm em}} \sim \frac{w^{\rm L,ind(d)}_{\rm abs,em}}{w^{\rm T,ind(d)}_{\rm abs,em}} \sim \frac{1}{2}\tanh(\tilde{\beta}\hbar\omega_{ji}/2)\,{\rm Re}\,\hat{\varepsilon}\,{\rm Re}\,\hat{\mu}\,\frac{\hbar^2\omega_{ji}^2}{m_{\rm t}^2c^4}\,. \tag{5.23}$$

The transition rates derived in this section for longitudinal radiation in Fermi statistics are evidently quite similar to those for transversal radiation in Bose statistics discussed in Section 4; had we quantized the transversal tachyonic modes in Fermi–Dirac statistics, this would have only affected the transversal absorption rates by a factor $\tanh(\tilde{\beta}\hbar\omega_{ji}/2)$.

Next we derive the longitudinal ionization cross section, cf. (4.32)–(4.34). The same procedures as outlined after (4.31) lead to, cf. (4.33) and (5.12),

$$w^{\rm L,ind}_{\rm abs} \sim \frac{1}{\pi}\frac{n_k}{L^3}\frac{q^2}{4\pi}\frac{k_{\rm e}\alpha_k^{\rm L^2}}{mc^2\hbar}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t} \times\left|\int {\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\nabla u_0\,{\rm d}\mathbf{x} - \frac{{\rm i}c^2k}{\omega_{\rm R}\,{\rm Re}\,\hat{\varepsilon}\,{\rm Re}\hat{\mu}}\frac{m}{\hbar}\int {\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}u_0\,{\rm d}\mathbf{x}\right|^2 {\rm d}\Omega_{\rm e}\,. \tag{5.24}$$

Dividing this by the flux density, $|\mathbf{v}_{\rm gr}|n_k/L^3$, cf. (4.34), we find

$${\rm d}\sigma^{\rm L} \sim \frac{1}{2\pi}\frac{q^2}{4\pi}\frac{1}{{\rm Re}\,\hat{\varepsilon}}\frac{k_{\rm e}k}{mc^2}\frac{m^2}{m_{\rm t}^2}\frac{{\rm e}^{-\omega_{\rm I}t}\sinh(\omega_{\rm I}t)}{\omega_{\rm I}t} \times\left(1-\boldsymbol{\varepsilon}_{\mathbf{k},3}\cdot\mathbf{k}_{\rm e}\frac{1}{k}\frac{\hbar\omega_{\rm e}}{mc^2}{\rm Re}\,\hat{\mu}\,{\rm Re}\,\hat{\varepsilon}\right)^2\left|\int {\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}u_0\,{\rm d}\mathbf{x}\right|^2{\rm d}\Omega_{\rm e}\,. \tag{5.25}$$

The second term in the parentheses should be taken into account, despite $\omega_{\rm R}\approx\omega_{\rm e}\ll mc^2/\hbar$, since $k_{\rm e}/k\gg 1$ is possible and $\omega_{\rm R}\gg\omega_0$ is needed for the plane wave approximation. Also compare in this regard the discussion of the maxima of the tansversal cross section in Ref. [14].

Next we discuss the equilibrium mechanics of longitudinal tachyonic black-body radiation, $\hat{\varepsilon}=\hat{\mu}=1$, $\omega_{\rm I}=0$; the thermodynamic formalism for the superluminal transversal modes has already been studied in Refs. [14,18]. The peak of the fermionic spectral density $\rho_{\rm L}(\omega)$ in (5.5) depends on the tachyon mass. Defining $x := \beta h\nu$ and $\gamma := \beta m_{\rm t}c^2$, we find the location of this peak by solving

$$\frac{x}{1+{\rm e}^{-x}} = 2 + \frac{x^2}{x^2+\gamma^2}\,. \tag{5.26}$$

As for the cosmic tachyon background, $\gamma\approx 9.1\times 10^6$ and $\beta h\approx 1.76\times 10^{-11}$ s, based on $m_{\rm t}\approx 2.15$ keV$/c^2$ and $kT\approx 2.35\times 10^{-4}$ eV. Since $x(\gamma\to\infty)\approx 2.218$, the longitudinal radiation has a frequency peak at $\nu_t^L \approx 126\,\mathrm{GHz}$ [$5.2\times10^{-4}\,\mathrm{eV}$ or $h\nu_t^L/(m_t c^2) \approx 2.4\times10^{-7}$], which is rather close to the peak of the photon density at $\nu_{ph} \approx 160\,\mathrm{GHz}$ ($6.6\times10^{-4}\,\mathrm{eV}$). The transversal tachyon spectrum is peaked at $\nu_t^T \approx 90.6\,\mathrm{GHz}$ or $3.7\times10^{-4}\,\mathrm{eV}$, cf. Ref. [18]. In dipole approximation, cf. (5.23),

$$\frac{w^L}{w^T} \sim \frac{1}{2}\tanh\left(\frac{\beta h\nu}{2}\right)\frac{h^2\nu^2}{m_t^2c^4}, \quad \frac{w^T}{w^{ph}} \sim \frac{\alpha_t}{\alpha_e}\,\frac{\sqrt{h^2\nu^2+m_t^2c^4}}{h\nu}\,, \tag{5.27}$$

where $\alpha_t/\alpha_e \approx 1.4\times10^{-11}$ is the ratio of tachyonic and electric fine structure constants. Accordingly, at $T \approx 2.725\,\mathrm{K}$, we find $w^L/w^T(\nu_t^L) \approx 1.45\times10^{-14}$ and $w^T/w^{ph}(\nu_t^L) \approx 5.8\times10^{-5}$, and comparable ratios for ν_t^T and ν_{ph}. Apparently, ν_t^L also lies well within the core of the photon and transversal tachyon distributions. In the high-temperature limit, we find from (5.26) $x(\gamma \to 0) \approx 3.131$. The solution of (5.26) varies only moderately over the whole temperature range, and the same holds for the maximum of the transversal bosonic distribution, cf. Ref. [18]. A frequency defined by $\beta h\nu_0(T) = x \approx 2.218$ lies for any temperature in the bulk of the enumerated distributions. Thus the location of the bulk of the spectral densities scales linearly with temperature, and Wien's displacement law is more or less recovered.

With these preparations, we can readily show that both the longitudinal and transversal quanta of the cosmic tachyon background have reached equilibrium at a rather early stage. Primordial nucleosynthesis requires a photonic equilibrium distribution at $kT \approx 1$ MeV, corresponding to a cosmic age of 1 s. At this temperature, we find $h\nu_0 \approx 2.2\,\mathrm{MeV}$, and the ratios $w^L/w^T(\nu_0) \approx 2.6\times10^5$ and $w^T/w^{ph}(\nu_0) \approx 1.4\times10^{-11}$. With increasing temperature, w^L/w^T gets quickly larger and overpowers w^T/w^{ph} in their product. Thus one can assume that the longitudinal tachyon background has reached equilibrium within the first second, even before the photon background did, due to its stronger interaction with subluminal matter. As for the transversal tachyon radiation, this happened at a much later stage, but well within 10^{11} s (assuming a linear space expansion in this early epoch), the more so as w^T/w^{ph} increases in time. In the present low-temperature regime, the ratios indicated after (5.27) hold, which makes the longitudinal fermionic radiation much harder to observe than the transversal boson background.

We turn to atoms in equilibrium with tachyon radiation. The dipole approximation (5.27) is the same for induced and spontaneous emission and absorption. We identify the Ly-α lines of hydrogen (10.2 eV) with the core frequency ν_0 as defined above, corresponding to a temperature of $kT(\nu_0) \approx 4.6\,\mathrm{eV}$. We so find from (5.27) $w^L/w^T(\nu_0) \approx 5.6\times10^{-6}$ and $w^T/w^{ph}(\nu_0) \approx 3.0\times10^{-9}$. The Ly-$\alpha_1$ transition in hydrogenic uranium ($\nu_0 \approx 0.23$ MeV) results in a bulk temperature of $kT(\nu_0) \approx 0.1$ MeV. (At this temperature, the frequency ν_0 lies in the core of the photonic and the two tachyonic spectral distributions.) In this case we find $w^L/w^T(\nu_0) \approx 2.9\times10^3$, but a very small $w^T/w^{ph}(\nu_0) \approx 1.4\times10^{-11}$. The chances to detect transversal tachyons improve in the low frequency fringe, due to the different frequency scaling of the photon and transversal tachyon distributions, cf. Ref. [18].

Finally we assemble the equations of state and the various thermodynamic variables for the longitudinal fermionic quanta. The subsequent high- and low-temperature expansions are derived in the Appendix, cf. (A.27) and (A.28). For $T \to 0$, we find in lowest order,

$$U \sim 6\pi\zeta(3)\,\frac{m_{\rm t}}{h^3 c}\,(kT)^3 V, \quad P \sim 3\pi\zeta(3)\,\frac{m_{\rm t}}{h^3 c}\,(kT)^3\,,$$

$$S \sim 9\pi\zeta(3)\,\frac{m_{\rm t}}{h^3 c}\,k^3 T^2 V, \quad c_V \sim 18\pi\zeta(3)\,\frac{m_{\rm t}}{h^3 c}\,k^3 T^2 V, \quad \frac{c_P}{c_V} \sim \frac{3}{2}\,,$$

$$N \sim \frac{\pi^3}{3}\,\frac{m_{\rm t}}{h^3 c}(kT)^2 V, \quad U \sim \frac{18\zeta(3)}{\pi^2} kTN \sim 2PV, \quad S \sim \frac{1}{2} c_V \sim \frac{3}{2}\,\frac{U}{T}\,. \tag{5.28}$$

(We have dropped the subscript L.) The caloric and thermal equations get independent of the tachyon mass in this limit. The high-temperature limit reads in leading order

$$U \sim \frac{7\pi^5}{30}\,\frac{1}{h^3 c^3}\,(kT)^4 V, \quad P \sim \frac{7\pi^5}{90}\,\frac{1}{h^3 c^3}\,(kT)^4\,,$$

$$S \sim \frac{14\pi^5}{45}\,\frac{1}{h^3 c^3}\,k^4 T^3 V, \quad c_V \sim \frac{14\pi^5}{15}\,\frac{1}{h^3 c^3}\,k^4 T^3 V, \quad \frac{c_P}{c_V} \sim \frac{4}{3}\,,$$

$$N \sim 6\pi\zeta(3)\frac{1}{h^3 c^3}(kT)^3 V, \quad U \sim \frac{7\pi^4}{180\zeta(3)} kTN \sim 3PV, \quad S \sim \frac{1}{3} c_V \sim \frac{4}{3}\,\frac{U}{T}\,. \tag{5.29}$$

The tachyon mass does not enter in lowest order. The corresponding expansions for transversal tachyon radiation are very similarly structured, with moderately modified numerical constants [14], despite the different frequency scaling of the spectral densities in the Rayleigh–Jeans limit.

6. Conclusion

A new quantum statistics for superluminal radiation has been suggested, resulting in a positive definite Hamiltonian and a stable ground state. We have departed from the customary field theoretic quantization, and chosen more elementary statistical procedures adapted to the extremely small tachyonic fine structure constant, $q^2/(4\pi\hbar c) \approx 1.0 \times 10^{-13}$. The tiny coupling constant is key to the quantization of superluminal radiation fields, this has been overlooked hitherto, as there have not been quantitative estimates on the interaction strength of superluminal radiation with matter. The weak interaction renders systematic quantum field theoretic expansions academic, which are anyway marred by negative energies and unstable vacua [1–3]. In this paper we have used detailed equilibrium balancing to describe the tiny interaction of tachyons with subluminal matter.

The statistical quantization developed here is completely self-consistent, and we have shown that it works, being capable of quantitative predictions based on realistic interactions, extremely weak but not out of reach. In the following I summarize the main features of superluminal quantum statistics. Tachyonic quantum ensembles are neither bosonic nor fermionic, they are a mixture of both. Statistics does not apply to the field, but to its modes, and in the case of superluminal radiation, we are not bound by the spin-statistics theorem, which requires the same algebraic relations for all modes. This freedom in the choice of the commutation relations for the Fourier amplitudes is used to render the energy density positive definite, ensuring a stable ground state. The superluminal quanta, transversal bosons and longitudinal fermions, admit gray-body spectra in quasi-equilibrium, with an adiabatic time variation of the temperature due to energy dissipation.

This energy dissipation relates to the underlying space structure. A consistent statistics of superluminal quanta cannot be achieved in a relativistic spacetime, due to causality violation manifested in advanced wave modes. Superluminal quanta need a very different context, a permeable space, the ether [15]. They interact with the refractive and absorptive ether, and this results in adiabatic energy dissipation. The starting point for quantization is the time averaged energy-balance equation, relating the superluminal energy flux to the field energy and the dissipated energy, cf. Section 3. This energy-balance is quantized by defining algebraic relations for the Fourier amplitudes. The statistics is chosen in a way to turn both the field energy and the dissipated energy into positive definite hermitian operators, transversal modes are bosons, longitudinal ones satisfy Fermi statistics, cf. Sections 4 and 5. Once a positive definite Hamiltonian is established for the free superluminal modes, it is straightforward to balance emission and absorption rates, in this way arriving at Einstein coefficients which are adiabatically damped by the energy dissipation.

Spontaneous emission is in no way hampered by the exclusion principle, it applies to longitudinal fermionic quanta as well. The semiclassical arguments used in Ref. [14] are quite efficient to describe bosonic superluminal modes, but they also have their limits. The bosonic transition rates derived in second quantization (Section 4) coincide with those of Ref. [14], provided we drop the adiabatic damping factors in the Einstein coefficients; the semiclassical derivation given in Ref. [14] does not account for the absorptivity of the ether and the resulting energy dissipation. Also, the exclusion principle is beyond semiclassical mechanics, and so we didn't attempt to study the longitudinal fermionic modes in Ref. [14]. The black-body limit of the fermionic equations of state is derived in the Appendix; the complete set of fermionic equilibrium variables is given in (5.28) and (5.29), complementing the variables for the transversal bosonic modes listed in (5.36) of Ref. [14].

In this paper we studied the equilibrium mechanics of superluminal quanta. We have not discussed applications, apart from some estimates on the cosmic tachyon background. But there are a variety of systems where one can try to spot quantum tachyons by means of the transition rates calculated in Sections 4 and 5. Estimates of tachyonic ionization cross sections of Rydberg atoms are given in Ref. [14], the effect of tachyon radiation on Lamb shifts in hydrogenic systems and on hyperfine intervals is studied in Ref. [18]. Superluminal cyclotron radiation in planetary magnetospheres, and tachyonic synchrotron radiation and inverse Compton scattering in the magnetic fields of supernova remnants will be discussed elsewhere.

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Appendix A. Equilibrium mechanics of superluminal fermionic modes

We derive the high- and low-temperature expansions of the internal and free energy as well as the number density, based on the Fermi–Dirac spectral function (5.5) of the longitudinal modes in the black-body limit. The thermodynamic formalism for the transversal modes in Bose–Einstein statistics has already been worked out in Ref. [14], and the longitudinal modes do not really require new calculations but some reassembling.

We start with the internal energy, cf. (5.4) (with $\hat{\varepsilon} = \hat{\mu} = 1$, $\omega_{\rm I} = 0$),

$$U_{\rm L}/V = 4\pi m_{\rm t}^4 c^5 h^{-3} \hat{U}(\alpha), \quad \hat{U}(\alpha) := \int_0^\infty \frac{\sqrt{1+x^2}}{e^{\alpha x}+1} x^2 \, dx \,, \tag{A.1}$$

where $\alpha := m_{\rm t}c^2/(kT)$. The asymptotic low-temperature limit is easily calculated via Watson's Lemma, by replacing the root in (A.1) by its series expansion,

$$\begin{aligned}\hat{U}(\alpha) &\sim \sum_{n=0}^{\infty} \binom{1/2}{n} \int_0^\infty \frac{x^{2n+2}\,dx}{e^{\alpha x}+1} \\ &= \frac{1}{\alpha^3} \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{\alpha^{2n}} (1 - 2^{-2n-2}) \Gamma(2n+3) \zeta(2n+3) \,. \end{aligned}\tag{A.2}$$

In the high-temperature limit, a systematic convergent expansion of $\hat{U}(\alpha)$ is obtained by means of the Euler series ($x < \pi$)

$$\frac{1}{e^x+1} = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(1-2^{2k+2})}{\Gamma(2k+3)} B_{2k+2} x^{2k+1} \,. \tag{A.3}$$

We split the integral (A.1) into $\hat{U}(\alpha) = \hat{U}_0 + \hat{U}_\infty$,

$$\hat{U}_0 := \frac{1}{\alpha^4} \int_0^\delta \frac{\sqrt{x^2+\alpha^2}}{e^x+1} x^2 \, dx, \qquad \hat{U}_\infty := \frac{1}{\alpha^4} \sum_{n=0}^{\infty} \binom{1/2}{n} \alpha^{2n} \int_\delta^\infty \frac{x^{3-2n}\,dx}{e^x+1} \,, \tag{A.4}$$

and choose $\alpha < \delta < \pi$. Series (A.3) converges absolutely in $[0, \delta]$; we substitute it into $\hat{U}_0$ and interchange summation and integration, arriving so at

$$\hat{U}_0 = \hat{U}_0^{(1)} + \hat{U}_0^{(2)}, \quad \hat{\alpha} := \alpha/\delta < 1 \,,$$

$$\hat{U}_0^{(1)} := \frac{1}{2}\frac{1}{\hat{\alpha}^4}\int_0^1 \sqrt{x^2+\hat{\alpha}^2}x^2\,\mathrm{d}x = \frac{1}{8}\frac{1+\hat{\alpha}^2/2}{\hat{\alpha}^4}\sqrt{1+\hat{\alpha}^2} - \frac{1}{16}\log\frac{1+\sqrt{1+\hat{\alpha}^2}}{\hat{\alpha}}\,,$$

$$\hat{U}_0^{(2)} := \frac{1}{\hat{\alpha}^4}\sum_{k=0}^{\infty}\frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)}\,\delta^{2k+1}\int_0^1 x^{2k+3}\sqrt{x^2+\hat{\alpha}^2}\,\mathrm{d}x\,. \tag{A.5}$$

The integrals in (A.5) represent hypergeometric functions [14], that is,

$$\int_0^1 x^{2k+3}\sqrt{x^2+\hat{\alpha}^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{4}\frac{\Gamma(k+2)(-)^k}{\Gamma(k+7/2)}\hat{\alpha}^{2k+5} - \frac{1}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n-1/2)(-)^n\hat{\alpha}^{2n}}{\Gamma(n+1)(2(k-n)+5)}\,. \tag{A.6}$$

Finally we insert (A.6) into (A.5), and interchange summations, $\hat{U}_0^{(2)} = \hat{U}_0^{(2)\mathrm{odd}} + \hat{U}_0^{(2)\mathrm{even}}$,

$$\hat{U}_0^{(2)\mathrm{odd}} := \frac{\sqrt{\pi}}{4}\alpha\sum_{k=0}^{\infty}\frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)}\frac{\Gamma(k+2)(-)^k}{\Gamma(k+7/2)}\alpha^{2k}\,,$$

$$\hat{U}_0^{(2)\mathrm{even}} := \frac{1}{2\sqrt{\pi}}\frac{1}{\hat{\alpha}^4}\sum_{n=0}^{\infty}c_n(\delta)\,\frac{\Gamma(n-1/2)(-)^{n+1}}{\Gamma(n+1)}\hat{\alpha}^{2n}\,, \tag{A.7}$$

$$c_n(\delta) := \sum_{k=0}^{\infty}\frac{(1-2^{2k+2})B_{2k+2}\delta^{2k+1}}{\Gamma(2k+3)(2(k-n)+5)}\,. \tag{A.8}$$

The series in (A.4)–(A.8) constitute the high-temperature expansion of the internal energy,

$$\hat{U}(\alpha) = \hat{U}_\infty + \hat{U}_0^{(1)} + \hat{U}_0^{(2)\mathrm{odd}} + \hat{U}_0^{(2)\mathrm{even}}\,. \tag{A.9}$$

The coefficients in this expansion must be independent of δ, which can easily be checked by means of (A.4) and the integral representation of the $c_n(\delta)$,

$$c_0(\delta) = \frac{1}{\delta^4}\int_0^{\delta}\frac{x^3\,\mathrm{d}x}{\mathrm{e}^x+1} - \frac{1}{8},\quad c_1(\delta) = \frac{1}{\delta^2}\int_0^{\delta}\frac{x\,\mathrm{d}x}{\mathrm{e}^x+1} - \frac{1}{4},$$

$$c_2(\delta) = \int_0^{\delta}\left(\frac{1}{\mathrm{e}^x+1} - \frac{1}{2}\right)\frac{\mathrm{d}x}{x}\,, \tag{A.10}$$

and similarly for $n > 2$, with more terms subtracted. The series in (A.4) and (A.7) converge for $\alpha < \pi$, which defines the convergence radius of the high-temperature expansion. I also remark, as an addendum to Eq. (4.19) of Ref. [14], where the internal energy of the transversal bosonic modes was studied, that $\gamma_0 = \frac{1}{2}(1-\log(2\pi))$. This constant is defined in Eq. (4.17) of Ref. [14] by an integral which can be evaluated in closed form as indicated.

The free energy can be handled quite similarly,

$$F_L/V = -(4\pi/3)m_t^4 c^5 h^{-3}\hat{F}(\alpha)\,,$$
$$\hat{F}(\alpha) := \frac{3}{\alpha}\int_0^\infty \sqrt{1+x^2}\,\log(1+e^{-\alpha x})\,x\,dx$$
$$= \int_0^\infty \frac{(1+x^2)^{3/2}}{e^{\alpha x}+1}\,dx - \frac{1}{\alpha}\log 2\,, \tag{A.11}$$

so that the low-temperature expansion reads

$$\hat{F}(\alpha) \sim \frac{1}{\alpha}\sum_{n=1}^\infty \binom{3/2}{n}\frac{1}{\alpha^{2n}}(1-2^{-2n})\Gamma(2n+1)\zeta(2n+1)\,. \tag{A.12}$$

The high-temperature expansion is again obtained by splitting the integral and expanding either nominator or denominator, $\hat{F}(\alpha) = \hat{F}_0 + \hat{F}_\infty - (1/\alpha)\log 2$,

$$\hat{F}_0 := \frac{1}{\alpha^4}\int_0^\delta \frac{(x^2+\alpha^2)^{3/2}}{e^x+1}\,dx, \quad \hat{F}_\infty := \frac{1}{\alpha^4}\sum_{n=0}^\infty \binom{3/2}{n}\alpha^{2n}\int_\delta^\infty \frac{x^{3-2n}\,dx}{e^x+1}\;; \tag{A.13}$$

$\hat{F}_0$ is calculated by means of the Euler expansion (A.3), $\hat{F}_0 = \hat{F}_0^{(1)} + \hat{F}_0^{(2)}$,

$$\hat{F}_0^{(1)} := \frac{1}{2}\frac{1}{\hat{\alpha}^4}\int_0^1 (x^2+\hat{\alpha}^2)^{3/2}\,dx$$
$$= \frac{1}{8}\frac{1}{\hat{\alpha}^4}\left(1+\frac{5}{2}\hat{\alpha}^2\right)\sqrt{1+\hat{\alpha}^2} + \frac{3}{16}\log\frac{1+\sqrt{1+\hat{\alpha}^2}}{\hat{\alpha}}\,,$$
$$\hat{F}_0^{(2)} := \frac{1}{\hat{\alpha}^4}\sum_{k=0}^\infty \frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)}\delta^{2k+1}\int_0^1 (x^2+\hat{\alpha}^2)^{3/2}x^{2k+1}\,dx\,. \tag{A.14}$$

The integrals in (A.14) admit the expansion [14]

$$\int_0^1 x^{2k+1}(x^2+\hat{\alpha}^2)^{3/2}\,dx = \frac{3\hat{\alpha}^{2k+5}}{8\sqrt{\pi}}\Gamma(k+1)\Gamma(-k-5/2)$$
$$+ \frac{3}{4\sqrt{\pi}}\sum_{n=0}^\infty \frac{\Gamma(n-3/2)(-)^n\hat{\alpha}^{2n}}{\Gamma(n+1)(2(k-n)+5)}\,, \tag{A.15}$$

and by combining (A.14) and (A.15), we obtain $\hat{F}_0^{(2)} = \hat{F}_0^{(2)\mathrm{odd}} + \hat{F}_0^{(2)\mathrm{even}}$,

$$\hat{F}_0^{(2)\mathrm{odd}} := \frac{3\alpha}{8\sqrt{\pi}}\sum_{k=0}^\infty \frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)}\Gamma(k+1)\Gamma(-k-5/2)\alpha^{2k}\,,$$
$$\hat{F}_0^{(2)\mathrm{even}} := \frac{3}{4\sqrt{\pi}}\frac{1}{\hat{\alpha}^4}\sum_{n=0}^\infty c_n(\delta)\frac{\Gamma(n-3/2)(-)^n}{\Gamma(n+1)}\hat{\alpha}^{2n}\,. \tag{A.16}$$

The coefficients $c_n(\delta)$ are the same as in (A.8) and (A.10). [Apart from an integration constant, $\hat{F}(\alpha)$ follows from $\hat{U}(\alpha)$ via term by term integration, $3\hat{U} = -\partial(\alpha\hat{F})/\partial\alpha$.] The high-temperature expansion of the free energy reads, cf. (A.13), (A.14) and (A.16),

$$\hat{F}(\alpha) = \hat{F}_0^{(1)} + \hat{F}_0^{(2)\mathrm{even}} + \hat{F}_0^{(2)\mathrm{odd}} + \hat{F}_\infty - (1/\alpha)\log 2\,. \tag{A.17}$$

We turn to the fermionic number density,

$$N_\mathrm{L}/V = 4\pi m_\mathrm{t}^3 c^3 h^{-3}\hat{N}(\alpha), \quad \hat{N}(\alpha) := \int_0^\infty \frac{\sqrt{x^2+1}}{\mathrm{e}^{\alpha x}+1}\, x\,\mathrm{d}x\,, \tag{A.18}$$

which is not an independent variable, as the chemical potential vanishes. The low-temperature expansion reads

$$\hat{N}(\alpha) \sim \frac{1}{\alpha^2}\sum_{n=0}^{\infty}\binom{1/2}{n}\frac{1}{\alpha^{2n}}(1-2^{-2n-1})\Gamma(2n+2)\zeta(2n+2)\,, \tag{A.19}$$

and the high-temperature limit is likewise found completely analogously to the foregoing. At first we write $\hat{N}(\alpha) = \hat{N}_0 + \hat{N}_\infty$,

$$\hat{N}_0 := \frac{1}{\alpha^3}\int_0^\delta \frac{\sqrt{x^2+\alpha^2}}{\mathrm{e}^x+1}x\,\mathrm{d}x, \quad \hat{N}_\infty := \frac{1}{\alpha^3}\sum_{n=0}^{\infty}\binom{1/2}{n}\alpha^{2n}\int_\delta^\infty \frac{x^{2-2n}\,\mathrm{d}x}{\mathrm{e}^x+1}\,, \tag{A.20}$$

and $\hat{N}_0$ is then further split into $\hat{N}_0 = \hat{N}_0^{(1)} + \hat{N}_0^{(2)}$,

$$\hat{N}_0^{(1)} := \frac{1}{2}\frac{1}{\hat{\alpha}^3}\int_0^1 \sqrt{x^2+\hat{\alpha}^2}\,x\,\mathrm{d}x = \frac{1}{6\hat{\alpha}^3}(1+\hat{\alpha}^2)^{3/2} - \frac{1}{6}\,,$$

$$\hat{N}_0^{(2)} := \frac{1}{\hat{\alpha}^3}\sum_{k=0}^{\infty}\frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)}\,\delta^{2k+1}\int_0^1 \sqrt{x^2+\hat{\alpha}^2}\,x^{2k+2}\,\mathrm{d}x\,. \tag{A.21}$$

The series expansion of the integrals in (A.21) is

$$\begin{aligned}\int_0^1 & x^{2k+2}\sqrt{x^2+\hat{\alpha}^2}\,\mathrm{d}x \\ &= \frac{(-)^{k+1}}{4\sqrt{\pi}}\frac{\Gamma(k+3/2)}{\Gamma(k+3)}\,\hat{\alpha}^{2k+4}(\psi(k+3)-\psi(k+3/2)-2\log\hat{\alpha}) \\ &\quad - \frac{1}{4\sqrt{\pi}}\sum_{\substack{n=0\\ n\neq k+2}}^{\infty}\frac{\Gamma(n-1/2)(-)^n\hat{\alpha}^{2n}}{\Gamma(n+1)(k-n+2)}\,.\end{aligned} \tag{A.22}$$

ψ is the logarithmic derivative of the gamma function, elementary for (half-)integers. It enters here by a limit procedure, $k \to k+\varepsilon$, needed to circumvent the poles in the hypergeometric functions defined by the integrals, cf. Ref. [14]. Thus, $\hat{N}_0^{(2)} = \hat{N}_0^{(2)(\mathrm{a})} +$

$\hat{N}_0^{(2)(b)}$,

$$\hat{N}_0^{(2)(a)} := \frac{\alpha}{4\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1-2^{2k+2})B_{2k+2}}{\Gamma(2k+3)} \frac{\Gamma(k+3/2)(-)^{k+1}}{\Gamma(k+3)} \alpha^{2k} \times (\psi(k+3) - \psi(k+3/2) - 2\log\hat{\alpha}) ,$$

$$\hat{N}_0^{(2)(b)} := \frac{1}{2\sqrt{\pi}} \frac{1}{\hat{\alpha}^3} \sum_{n=0}^{\infty} \tilde{c}_n(\delta) \frac{\Gamma(n-1/2)(-)^{n+1}}{\Gamma(n+1)} \hat{\alpha}^{2n} , \tag{A.23}$$

$$\tilde{c}_n(\delta) := \sum_{\substack{k=0 \\ k \neq n-2}} \frac{(1-2^{2k+2})B_{2k+2}\delta^{2k+1}}{\Gamma(2k+3)(2(k-n)+4)} . \tag{A.24}$$

The coefficients (A.24) admit integral representations such as, cf. (A.10),

$$\tilde{c}_0(\delta) = \frac{1}{\delta^3} \int_0^{\delta} \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x + 1} - \frac{1}{6}, \quad \tilde{c}_1(\delta) = \frac{1}{\delta} \int_0^{\delta} \frac{\mathrm{d}x}{\mathrm{e}^x + 1} - \frac{1}{2} . \tag{A.25}$$

The high-temperature expansion of the particle density is thus, cf. (A.20), (A.21) and (A.23),

$$\hat{N}(\alpha) = \hat{N}_0^{(1)} + \hat{N}_0^{(2)(a)} + \hat{N}_0^{(2)(b)} + \hat{N}_\infty . \tag{A.26}$$

To summarize, we list the first two terms of the series expansions derived in this Appendix. In the low-temperature limit, we find from (A.2), (A.12) and (A.19),

$$\hat{U}(\alpha) \sim \frac{3}{2}\zeta(3)\frac{1}{\alpha^3}\left(1 + \frac{15}{2}\frac{\zeta(5)}{\zeta(3)}\frac{1}{\alpha^2} + \cdots\right) ,$$

$$\hat{F}(\alpha) \sim \frac{9}{4}\zeta(3)\frac{1}{\alpha^3}\left(1 + \frac{15}{4}\frac{\zeta(5)}{\zeta(3)}\frac{1}{\alpha^2} + \cdots\right) ,$$

$$\hat{N}(\alpha) \sim \frac{\pi^2}{12}\frac{1}{\alpha^2}\left(1 + \frac{7\pi^2}{20}\frac{1}{\alpha^2} + \cdots\right) , \tag{A.27}$$

with $\zeta(3) \approx 1.202$ and $\zeta(5) \approx 1.037$. The first two terms of the high-temperature expansions follow from (A.9), (A.17) and (A.26),

$$\hat{U}(\alpha) = \frac{7}{120}\frac{\pi^4}{\alpha^4}\left(1 + \frac{5}{7}\frac{\alpha^2}{\pi^2} + \cdots\right) ,$$

$$\hat{F}(\alpha) = \frac{7}{120}\frac{\pi^4}{\alpha^4}\left(1 + \frac{15}{7}\frac{\alpha^2}{\pi^2} + \cdots\right), \quad \hat{N}(\alpha) = \frac{3}{2}\zeta(3)\frac{1}{\alpha^3}\left(1 + \frac{\log 2}{3\zeta(3)}\alpha^2 + \cdots\right) . \tag{A.28}$$

All other fermionic variables, cf. (5.28) and (5.29), can be assembled from (A.1), (A.11) and (A.18).

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