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Linear Gravitational Waves and Electrodynamic Formalism in Cosmology

Roman Tomaschitz[1]

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The propagation of linear gravitational waves is studied in open and multiply connected Robertson–Walker cosmologies. In order for the group velocity of the gravitational wave packets to coincide with the speed of light, the linear wave equation must be conformally coupled. This opens the possibility of using the electromagnetic formalism. The gravitational analogue to the electromagnetic field tensor is introduced, and a tensorial counterpart to Maxwell's equations on the spacelike 3-slices is derived. The energy-momentum tensor for linear gravitational waves is constructed without averaging procedures, a strictly positive energy density is obtained, and it is shown that the overall energy of a gravitational pulse scales with the inverse of the expansion factor.

1. INTRODUCTION

The high degree of symmetry of the Robertson–Walker (RW) line element offers an approach to linear gravitational waves in close analogy to electrodynamics, a tensorial version of Maxwell's theory. The very great advantage of this cosmological background geometry is that a positive energy density can be derived for gravitational pulses without using the averaging procedures necessary in the standard theory of linearized gravitational waves.

The starting point of the standard theory is the linearization of the Ricci tensor. In the linear wave equation derived in this way one usually drops some curvature terms on the grounds of smallness considerations, on which I will not expand here. So the linear equations discussed in the literature differ somewhat, depending on the terms actually dropped (Isaacson , 1968; Misner *et al.*, 1973; Landau and Lifshitz, 1962). At any rate, they are, as

[1] Department of Physics, Hiroshima University, 1-3-1 Kagami-yama, Higashi-Hiroshima 724, Japan, and Inter-University Centre for Astronomy and Astrophysics, Ganeshkhind, Pune, 411 007, India.

linear equations, inconsistent either with the gauge conditions imposed or with the requirement that the wave packets move with the speed of light. In passing, we will demonstrate this very explicitly in RW cosmologies. The usual argument, that these inconsistencies are of higher order in certain heuristically defined expansion parameters, does not change anything, and they become a real obstacle if one studies gravitational waves in multiply connected RW cosmologies with noncommutative covering groups, as we will do here.

In this paper we choose a very different approach in the context of RW background geometries. We derive a linear wave equation which is (1) invariant with respect to infinitesimal coordinate transformations in the Minkowskian limit (i.e., when spatial curvature and expansion rate approach zero), (2) consistent with the gauge conditions (Lorentz condition, trace condition, and requirement that time–time and space–time components of the wave field are zero), and which is designed in such a way that (3) group and phase velocities are identical with the speed of light. This wave equation turns out to be conformally coupled. It can be derived from the Lagrangian $L_G = -\frac{1}{4}G_{\rho\mu\nu}G^{\rho\mu\nu}$, with $G_{\rho\mu\nu} = B_{\rho\nu;\mu} - B_{\rho\mu;\nu}$. Geodesic motion in a linear gravitational wave $B_{\mu\nu}$ is then defined with respect to the metric $g^{\mathrm{RW}}_{\mu\nu} + B_{\mu\nu}$.

From the structure of L_G it is not surprising that we can formulate the theory as a tensorial analogue to electrodynamics. In particular the energy density can be defined as $\rho = \frac{1}{2}(E^2 + B^2)$, with $E^2 := E_{ij}E^{ij}$ and $B^2 := B_{ij}B^{ij}$, where E_{ij} and B_{ij} are symmetric tensors on the 3-space of the RW geometry. We thus arrive at a well-defined positive energy density and a conservation law $a^4(\tau)\rho$ = const, with $a(\tau)$ the expansion factor in the background metric $g^{\mathrm{RW}}_{\mu\nu}$.

This paper is organized as follows. In Section 2 we derive a certain class of linear tensorial wave equations, discuss Lagrangians for them, show their compatibility with the gauge conditions usually imposed on gravitational waves, and point out their relation to the linearized Ricci tensor. In Section 3 we discuss the definition of the energy-momentum tensor, the positivity and conservation of energy, as well as the speed of wave propagation, and select on the basis of these criteria the wave equation suitable to describe freely propagating linear gravitational pulses in the RW geometry.

In Section 4 we discuss the time evolution of wave fields and the spectral theory of the wave equation in simply connected, open RW cosmologies with negatively curved spacelike slices. We also consider in this context the spin of wave fields and the composition of wave packets from the spectral elementary waves. In Section 5 we discuss wave propagation in RW cosmologies with multiply connected spacelike slices. We introduce automorphic tensor fields and sketch the orthogonality and completeness relations for the spectral elementary waves. In Section 6 we finally derive the tensorial analogue of

Maxwell's equations, at first manifestly covariantly, and then in terms of 3-tensors on the spacelike slices. In the Appendix we list some explicit formulas for the curvature tensor and covariant derivatives in RW geometries.

2. LAGRANGIANS FOR LINEAR GRAVITATIONAL WAVES

In Minkowski space the most general Lorentz invariant Lagrangian that leads to a linear wave equation for a symmetric tensor field $B_{\mu\nu}$ is

$$\begin{aligned} L = -\tfrac{1}{2}\{ & \kappa_1 B_{\mu\nu;\kappa} B^{\mu\nu;\kappa} + \kappa_2 B_{\mu\nu;\kappa} B^{\mu\kappa;\nu} + \kappa_3 B_{\kappa\nu}{}^{;\kappa} B_\lambda{}^{\nu;\lambda} + \kappa_4 B^\kappa_{\kappa;\nu} B^{\lambda;\nu}_\lambda \\ & + \kappa_5 B^\kappa_{\kappa;\nu} B^{\lambda\nu}{}_{;\lambda} + m_0 B_{\mu\nu} B^{\mu\nu} + m_{00} B^\kappa_\kappa B^\lambda_\lambda \} \end{aligned} \tag{2.1}$$

It is second order in the $B_{\mu\nu}$, and contains no higher than first-order derivatives. We obtain the wave equation

$$\begin{aligned} 0 &= -\left(\frac{\delta L}{\partial B^{\mu\nu}{}_{;\gamma}}\right)_{;\gamma} + \frac{\delta L}{\partial B^{\mu\nu}} \\ &= \kappa_1 B_{\mu\nu;\lambda}{}^{;\lambda} + \tfrac{1}{2}\kappa_2 (B_\mu{}^\kappa{}_{;\nu;\kappa} + B_\nu{}^\kappa{}_{;\mu;\kappa}) + \tfrac{1}{2}\kappa_3 (B_{\kappa\mu}{}^{;\kappa}{}_{;\nu} + B_{\kappa\nu}{}^{;\kappa}{}_{;\mu}) \\ &\quad + \kappa_4 B^\kappa_{\kappa;\gamma}{}^{;\gamma} g_{\mu\nu} + \tfrac{1}{2}\kappa_5 (B^\kappa_{\kappa;\mu;\nu} + B^{\kappa\lambda}{}_{;\lambda;\kappa} g_{\mu\nu}) - m_0 B_{\mu\nu} - m_{00} B^\kappa_\kappa g_{\mu\nu} \end{aligned} \tag{2.2}$$

We wrote (2.1) and (2.2) in terms of covariant derivatives, and did not interchange derivatives. So these equations hold true in an arbitrary curved space. L is then the most general covariant Lagrangian which does not contain curvature terms (i.e., the Riemann curvature tensor and its contractions).

In Minkowski space, $g_{\mu\nu} = \text{diag}(-1,1,1,1)$, we require that the wave equation (2.2) is invariant with respect to gauge transformations $B_{\mu\nu} \to B_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, with an arbitrary vector field ξ_μ. In order that $B_{\mu\nu} := \xi_{\mu,\nu} + \xi_{\nu,\mu}$ is a solution of (2.2) we must have $\kappa_2 + \kappa_3 = -2\kappa_1$, $\kappa_5 = 2\kappa_1$, $\kappa_4 = -\kappa_1$, and $m_0 = m_{00} = 0$. We assume these relations also in a curved space; cf. Remark 1 below. Clearly, $\kappa_1 \neq 0$, and we may choose, without loss of generality, $\kappa_1 = 1$. The wave equation (2.2) can now be written as

$$\begin{aligned} B_{\mu\nu;\lambda}{}^{;\lambda} - (B_{\kappa\mu}{}^{\kappa}{}_{;\nu} + B_{\kappa\nu}{}^{;\kappa}{}_{;\mu}) + B_\kappa{}^\kappa{}_{;\mu;\nu} + \kappa_2 B_{\kappa\lambda} R^\kappa{}_{\mu\nu}{}^\lambda \\ + \tfrac{1}{2}\kappa_2 (R_{\mu\lambda} B^\lambda{}_\nu + R_{\nu\lambda} B^\lambda{}_\mu) = 0 \end{aligned} \tag{2.3}$$

We interchanged here covariant derivatives,

$$B_{\mu\lambda;\nu}{}^{;\lambda} + B_{\nu\lambda;\mu}{}^{;\lambda} \equiv B_{\mu\lambda}{}^{;\lambda}{}_{;\nu} + B_{\nu\lambda}{}^{;\lambda}{}_{;\mu} + 2B_{\kappa\lambda} R^\kappa{}_{\mu\nu}{}^\lambda + R_{\mu\lambda} B^\lambda{}_\nu + R_{\nu\lambda} B^\lambda{}_\mu \tag{2.4}$$

[with sign conventions for the Riemann and Ricci tensors as in Misner *et al.* (1973) and Landau and Lifshitz (1962)], and used the contraction of equation (2.2),

$$B_{\mu}{}^{\mu}{}_{;\lambda}{}^{;\lambda} = B_{\mu\lambda}{}^{;\mu;\lambda} \tag{2.5}$$

Remarks. 1. In a curved space, $B_{\mu\nu} := \xi_{\mu;\nu} + \xi_{\nu;\mu}$ is no longer a solution of the wave equation (2.3). This gauge invariance is only recovered in the limit of vanishing curvature; in a RW cosmology this is the case in the limit of slow expansion, $\dot{a}/a \approx 0$, $\ddot{a}/a \approx 0$, and large spatial curvature radius $a(\tau)R$; cf. the Appendix.

2. For $\kappa_2 = -2$, the left side of equation (2.3) is $-2R^{\text{lin}}_{\mu\nu}$, where $R^{\text{lin}}_{\mu\nu}$ is the linearized Ricci tensor with the RW metric as background and $B_{\mu\nu}$ as the linearized fluctuation. We will later see that only the value $\kappa_2 = -1$ corresponds to conformal coupling and a wave propagation at the speed of light.

Let us next consider the Lagrangian

$$L_G := -\tfrac{1}{4}G_{\kappa\mu\nu}G^{\kappa\mu\nu} - \tfrac{1}{2}\alpha R_{\mu\nu}B^{\mu\lambda}B^{\nu}_{\lambda} - \tfrac{1}{2}\beta R_{\mu\kappa\lambda\nu}B^{\mu\nu}B^{\kappa\lambda} \tag{2.6}$$

with

$$G_{\kappa\mu\nu} := B_{\kappa\nu;\mu} - B_{\kappa\mu;\nu} \tag{2.7}$$

This tensor, which is the key to the electrodynamic formalism, was introduced in Fierz (1939). L_G is evidently not of the form (2.1), because it contains curvature terms. We have then as Lagrange equations

$$\begin{aligned} 0 &= -\left(\frac{\delta L_G}{\delta B^{\mu\nu}{}_{;\gamma}}\right)_{;\gamma} + \frac{\delta L_G}{\delta B^{\mu\nu}} \\ &= B_{\mu\nu;\lambda}{}^{;\lambda} - \tfrac{1}{2}(B_{\lambda\mu;\nu}{}^{;\lambda} + B_{\lambda\nu;\mu}{}^{;\lambda}) - \tfrac{1}{2}\alpha(R_{\mu\lambda}B^{\lambda}{}_{\nu} + R_{\nu\lambda}B^{\lambda}{}_{\mu}) - \beta B^{\kappa\lambda}R_{\mu\kappa\lambda\nu} \end{aligned} \tag{2.8}$$

Using equation (2.4), we obtain

$$\begin{aligned} &B_{\mu\nu;\lambda}{}^{;\lambda} - (1+\beta)B_{\kappa\lambda}R^{\kappa}{}_{\mu\nu}{}^{\lambda} - \tfrac{1}{2}(1+\alpha)(R_{\mu\lambda}B^{\lambda}{}_{\nu} + R_{\nu\lambda}B^{\lambda}{}_{\mu}) \\ &\quad - \tfrac{1}{2}(B_{\kappa\mu}{}^{;\kappa}{}_{;\nu} + B_{\kappa\nu}{}^{;\kappa}{}_{;\mu}) = 0 \end{aligned} \tag{2.9}$$

and with (2.7) we may write this as

$$G_{\rho\gamma\nu}{}^{;\gamma} - \tfrac{1}{2}G_{\gamma\rho\nu}{}^{;\gamma} - \tfrac{1}{2}\alpha(R_{\rho\lambda}B^{\lambda}{}_{\nu} + R_{\nu\lambda}B^{\lambda}{}_{\rho}) - \beta R_{\gamma\rho\nu\delta}B^{\gamma\delta} = 0 \tag{2.10}$$

$$G_{\gamma\rho\nu}{}^{;\gamma} \equiv R_{\rho\lambda}B^{\lambda}{}_{\nu} - R_{\nu\lambda}B^{\lambda}{}_{\rho} + B_{\nu\lambda}{}^{;\lambda}{}_{;\rho} - B_{\rho\lambda}{}^{;\lambda}{}_{;\nu} \tag{2.11}$$

Equations (2.1)–(2.11) were formulated in an arbitrary Riemannian space. From now on we assume a RW geometry, with a curvature tensor as defined in the Appendix.

We search for solutions of equation (2.9) which satisfy the trace and transversality conditions

$$B^{\alpha}_{\alpha} = 0 \tag{2.12}$$

$$B_{\mu\alpha}{}^{;\alpha} = 0 \tag{2.13}$$

In order that these conditions are consistent with the wave equation (2.9), we must require $\alpha = \beta$, which follows immediately by contracting equation (2.9). Moreover, we see that the wave equations (2.3) and (2.9) are identical if we put

$$\alpha = \beta, \qquad \kappa_2 = -(1 + \alpha) \tag{2.14}$$

and impose the subsidiary conditions (2.12) and (2.13). From now on we assume the identification (2.14).

In a general Riemannian space the subsidiary conditions (2.12) and (2.13) are not consistent with the wave equations (2.3) or (2.9), but in a RW geometry they are. Moreover, we can impose on the solutions of equation (2.9), in addition to (2.12) and (2.13),

$$B_{\mu 0} = 0 \tag{2.15}$$

To see that equations (2.9), (2.12), (2.13), and (2.15) are mutually consistent, one may use the explicit formulas for the curvature tensor in the Appendix, and the $\nu - \mu$ contraction of the identity

$$\begin{aligned} B_{\rho\nu;\mu;\alpha}{}^{;\alpha} \equiv {} & B_{\rho\nu;\alpha}{}^{;\alpha}{}_{;\mu} + 2B_{\kappa\nu;\alpha}R^{\kappa}{}_{\rho\mu}{}^{\alpha} + 2B_{\rho\kappa;\alpha}R^{\kappa}{}_{\nu\mu}{}^{\alpha} + B_{\rho\nu;\kappa}R^{\kappa}{}_{\mu} \\ & + B_{\kappa\nu}(R_{\mu}{}^{\kappa}{}_{;\rho} - R_{\mu\rho}{}^{;\kappa}) + B_{\rho\kappa}(R_{\mu}{}^{\kappa}{}_{;\nu} - R_{\mu\nu}{}^{;\kappa}) \end{aligned} \tag{2.16}$$

which holds true, by the way, in every Riemannian space. To obtain equation (2.16), one uses the commutation rules for covariant derivatives and the contracted Bianchi identity.

The Lagrangians (2.1) and (2.6) lead to identical wave equations under the indicated conditions. However, it turns out that L_G in (2.6) is the appropriate choice for the construction of the energy-momentum tensor.

3. ENERGY OF LINEAR GRAVITATIONAL WAVES

Throughout this section we assume a RW geometry as defined in the Appendix. In this context we define energy for wave packets satisfying the wave equation (2.9) and the subsidiary conditions (2.12), (2.13), and (2.15). In contrast to less symmetric space-time geometries, one need not resort to averaging procedures (e.g., Misner *et al.*, 1973) to define energy. Rather, we will proceed in quite a similar way as in electromagnetic theory.

Using standard Lagrange formalism, we start with the tensor

$$\hat{T}^{\mu}{}_{\nu} := -\frac{\delta L_G}{\delta B_{\alpha\beta;\mu}} B_{\alpha\beta;\nu} + g_{\mu\nu} L_G = -G_{\alpha\beta\mu} B^{\alpha\beta}{}_{;\nu} + g_{\mu\nu} L_G \tag{3.1}$$

with L_G as in (2.6) [$\beta = \alpha$; cf. (2.14)].

To symmetrize $\hat{T}_{\mu\nu}$, we add a divergence

$$T'_{\mu\nu} := (G^{\alpha\beta}{}_{\mu} B_{\alpha\nu})_{;\beta} = G_{\alpha\beta\mu} B^{\alpha}{}_{\nu}{}^{;\beta} + \tfrac{1}{2}(G^{\rho}{}_{\beta\mu}{}^{;\beta} B_{\rho\nu} + G^{\rho}{}_{\beta\nu}{}^{;\beta} B_{\rho\mu}) \tag{3.2}$$

[The tensor $G^{\rho}{}_{\beta\mu}{}^{;\beta} B_{\rho\nu}$ is symmetric, which follows from the wave equation (2.10) and the subsidiary conditions.] Under these conditions we have

$$T'_{\mu\nu}{}^{;\mu} = B_{\alpha\mu;\beta} B^{\alpha\lambda} R_{\lambda\nu}{}^{\beta}{}_{\mu} \tag{3.3}$$

We define the energy-momentum tensor for solutions of the wave equation (2.9) which satisfy the three subsidiary conditions as

$$\begin{aligned} T_{\mu\nu} = \hat{T}_{\mu\nu} + T'_{\mu\nu} = {} & G_{\alpha\beta\mu} G^{\alpha\beta}{}_{\nu} + \tfrac{1}{2}(G_{\rho\beta\mu}{}^{;\beta} B^{\rho}{}_{\nu} + G_{\rho\beta\nu}{}^{;\beta} B^{\rho}{}_{\mu}) \\ & - g_{\mu\nu}(\tfrac{1}{4} G_{\kappa\alpha\beta} G^{\kappa\alpha\beta} + \tfrac{1}{2}\alpha R_{\sigma\rho} B_{\alpha}{}^{\sigma} B^{\alpha\rho} + \tfrac{1}{2}\alpha R_{\kappa\alpha\beta\lambda} B^{\kappa\lambda} B^{\alpha\beta}) \end{aligned} \tag{3.4}$$

By means of the field equations (2.10), the subsidiary conditions, and the explicit formulas for the curvature tensor in the Appendix, we may write (3.4) as

$$T_{\mu\nu} = T^{G}_{\mu\nu} + T^{B}_{\mu\nu} \tag{3.5}$$

$$T^{G}_{\mu\nu} := G_{\alpha\beta\mu} G^{\alpha\beta}{}_{\nu} - \tfrac{1}{4} g_{\mu\nu} G_{\kappa\alpha\beta} G^{\kappa\alpha\beta} \tag{3.6}$$

$$T^{B}_{00} := \tfrac{1}{2} c^{-2} \alpha \left(\frac{-3}{R^2 a^2} + \frac{1}{c^2}\frac{\ddot{a}}{a} + 3\frac{1}{c^2}\frac{\dot{a}^2}{a^2} \right) B_{ab} B^{ab} \tag{3.7}$$

$$T^{B}_{mn} := \alpha \left(\frac{-3}{R^2 a^2} + \frac{1}{c^2}\frac{\ddot{a}}{a} + 3\frac{1}{c^2}\frac{\dot{a}^2}{a^2} \right) (B_{ml} B^{l}{}_{n} - \tfrac{1}{2} g_{mn} B_{ab} B^{ab}) \tag{3.8}$$

$$T^{B}_{m0} := 0 \tag{3.9}$$

The divergence $T_{\mu\nu}{}^{;\mu}$ is straightforward to calculate. It is convenient to replace $G_{\rho\beta\nu}{}^{;\beta} B^{\rho}{}_{\mu}$ by $G_{\rho\beta\mu}{}^{;\beta} B^{\rho}{}_{\nu}$ and to eliminate $G_{\rho\beta\mu}{}^{;\beta}$ by means of the field equations (2.10) before differentiating equation (3.4). Note in particular that

$$G_{\gamma\rho\nu}{}^{;\gamma} = 0 \tag{3.10}$$

$$G_{\alpha\beta}{}^{\mu} G^{\alpha\beta}{}_{\nu;\mu} - \tfrac{1}{2} G_{\alpha\beta\mu;\nu} G^{\alpha\beta\mu} = 0 \tag{3.11}$$

because of the subsidiary conditions and the special structure of the curvature tensor in a RW geometry. We finally obtain, using the contracted Bianchi identity, $R^{\mu}{}_{\delta\gamma\rho;\mu} = R_{\delta\rho;\gamma} - R_{\delta\gamma;\rho}$,

$$T_{\mu\nu}{}^{;\mu} = -\tfrac{1}{2}\alpha(R^{\alpha\beta}{}_{;\nu}B_{\alpha}{}^{\rho}B_{\beta\rho} + R_{\kappa\alpha\beta\lambda;\nu}B^{\kappa\lambda}B^{\alpha\beta}) \tag{3.12}$$

and thus

$$T_{\mu 0}{}^{;\mu} = -\tfrac{1}{2}\alpha B_{mn}B^{mn}\left[6\,\frac{\dot a}{a}\left(\frac{1}{R^2a^2} + \frac{1}{c^2}\frac{\ddot a}{a} - \frac{1}{c^2}\frac{\dot a^2}{a^2}\right) + \frac{1}{c^2}\frac{d}{d\tau}\frac{\ddot a}{a}\right] \tag{3.13}$$

$$T_{\mu n}{}^{;\mu} = 0 \tag{3.14}$$

If $\alpha = 0$, or if the Riemann tensor is covariantly constant, $R_{\alpha\beta\gamma\delta;\mu} = 0$, then $T_{\mu\nu}$ satisfies the differential conservation law $T_{\mu\nu}{}^{;\mu} = 0$.

Remark. RW cosmologies that admit a ten-parameter group of continuous symmetries (and in particular boosts that mix space and time) are characterized by an expansion factor which satisfies

$$\frac{1}{c^2}\frac{\ddot a}{a} = \frac{1}{c^2}\frac{\dot a^2}{a^2} - \frac{1}{a^2R^2}$$

and, as a consequence,

$$\frac{d}{d\tau}\frac{\ddot a}{a} = 0$$

We refer in the following to such cosmologies as of de Sitter/Minkowski type. Well-known examples are $a(\tau) = \sinh(cR^{-1}\tau)$, R positive; $a(\tau) = \cosh(c|R|^{-1}\tau)$, R imaginary; $a(\tau) = \sin(cR^{-1}\tau)$, R positive; $a(\tau) = \exp(\Lambda\tau)$, $R = \infty$, Λ is a constant; $a(\tau) = cR^{-1}\tau$, R positive (a flat 4-manifold); $a(\tau) = 1$, $R = \infty$. A positive curvature radius means negatively curved spacelike slices in our notation; cf. the Appendix. These geometries have a covariantly constant curvature tensor. The only RW cosmologies with this property are either maximally symmetric or static ($\dot a = 0$); cf. (A.3).

The structure of $T^{G}_{\mu\nu}$ in (3.6) is reminiscent of electromagnetic theory. We can introduce symmetric 3-tensors E_{mn}, H_{mn} on the 3-space of the RW geometry by defining

$$E_{mn} := c^{-1}G_{mn0} \tag{3.15}$$

$$H_m{}^n := \tfrac{1}{2}\gamma^{-1/2}\epsilon^{nij}G_{mij} \tag{3.16}$$

and inversely,

$$G_{mij} = \gamma^{1/2}\epsilon_{ijk}H_m{}^k \tag{3.17}$$

Latin indices run from 1 to 3, Greek ones from 0 to 3. The metric g_{ij} on the 3-space is defined at the beginning of the Appendix; γ denotes its determinant, and $\gamma^{1/2}\epsilon_{ijk}$ and $\gamma^{-1/2}\epsilon^{ijk}$ are the totally alternating co- and contravariant Levi-Civita tensors on the 3-space. It is understood that $G_{\rho\mu\nu}$ defined in (2.7) is composed of $B_{\mu\nu}$-fields which satisfy the subsidiary conditions (2.12), (2.13), and (2.15). With $E^2 := E_{ij}E^{ij}$ and $H^2 := H_{ij}H^{ij}$ we may then write

$$T^G_{00} = \tfrac{1}{2}c^2(E^2 + H^2) \tag{3.18}$$

$$T^G_{mn} = \tfrac{1}{2}(E^2 + H^2)g_{mn} - H_{ln}H^l{}_m - E_{ln}E^l{}_m \tag{3.19}$$

$$T^G_{m0} = c\gamma^{1/2}\epsilon_{mjk}H_l{}^jE^{lk} \tag{3.20}$$

Clearly, T^G_{00} is positive definite, but the energy density T_{00} of the wave field is composed of T^G_{00} *and* T^B_{00}, and T^B_{00} need not be positive for an arbitrary expansion factor; cf. (3.7). However, if $\alpha = 0$, which corresponds to a conformally coupled wave equation (cf. Section 4), we have $T^B_{\mu\nu} \equiv 0$, and thus a positive energy density $T_{00} = T^G_{00}$.

Note that in the Lagrangian (2.6) the α term ($\alpha = \beta$) explicitly reads

$$\tfrac{1}{2}\alpha(R_{\mu\nu}B^{\mu\rho}B^{\nu}_{\rho} + R_{\mu\alpha\beta\nu}B^{\alpha\beta}B^{\mu\nu}) = c^2T^B_{00} \tag{3.21}$$

with T^B_{00} as in (3.7). In the static case, $a = 1$, and with negative α this is just a mass term, $(mc/\hbar)^2 := -3\alpha/R^2$. In maximally symmetric background geometries T^B_{00} is also a constant multiple of $B_{ij}B^{ij}$; cf. the Remark after (3.14). For $\alpha = 1$ the wave equation (2.9) with the subsidiary conditions (2.12), (2.13), and (2.15) is equivalent to the linearized Einstein equations; cf. Remark 2 after equation (2.5) and the identification (2.14). A positive α, however, corresponds to an imaginary mass term (if $\dot{a} = 0$) and to superluminal velocities; see also the discussion after equation (4.25). This mass term is very tiny, and may be dropped in the linear approximation (Isaacson, 1968; Misner *et al.*, 1973; Landau and Lifshitz, 1962). In RW cosmology this means that we replace the linearized Einstein equations by the conformally coupled wave equation (2.9) ($\alpha = \beta = 0$). This linear wave equation ensures that linear gravitational waves propagate exactly at the speed of light and admit an energy-momentum tensor (3.18)–(3.20) in perfect analogy to vacuum electrodynamics. In the next two sections we will discuss the explicit construction of linear gravitational pulses by means of this wave equation.

4. COMPOSITION OF LINEAR GRAVITATIONAL WAVES

In this section we sketch the spectral theory of the wave equation (2.9) ($\alpha = \beta$) in a RW background metric with negatively curved spacelike slices. As in Section 3, we assume that the wave solutions of (2.9) satisfy the subsidiary conditions (2.12), (2.13), and (2.15).

To perform the time separation, we express the d'Alembertian in (2.9) in terms of covariant derivatives (||) on the 3-space; cf. (A.9). Taking into account the subsidiary conditions and the explicit formulas for the curvature tensor in the Appendix, we obtain from (2.9)

$$B_{mn\|k}{}^{\|k} - \frac{1}{c^2} B_{mn,0,0} + \frac{1}{c^2}\frac{\dot{a}}{a} B_{mn,0} + B_{mn}\left[3\frac{1}{R^2}\frac{1}{a^2}(1+\alpha) - \frac{1}{c^2}\frac{\dot{a}^2}{a^2}(1+3\alpha) + \frac{1}{c^2}\frac{\ddot{a}}{a}(1-\alpha)\right] = 0 \quad (4.1)$$

The comma followed by a zero means ordinary differentiation with respect to cosmic time τ. The space–time and time–time components of the wave equation vanish identically.

With the separation ansatz $B_{mn} =: \varphi(\tau)\hat{B}_{mn}$ we obtain from (4.1)

$$a^2\hat{B}_{mn\|k}{}^{\|k} + \frac{\lambda}{R^2}\hat{B}_{mn} = 0 \quad (4.2)$$

and

$$\varphi_{,0,0} - \frac{\dot{a}}{a}\varphi_{,0} + \varphi\left[\frac{c^2}{R^2a^2}(\lambda - 3 - 3\alpha) + \frac{\dot{a}^2}{a^2}(1+3\alpha) - \frac{\ddot{a}}{a}(1-\alpha)\right] = 0 \quad (4.3)$$

λ is the separation constant. Note that the 3-space metric g_{ij} scales with $a^2(\tau)$, so the Laplacian $\hat{B}_{mn\|k}{}^{\|k}$ on the 3-space scales with $a^{-2}(\tau)$; cf. (A.11).

We define now $\Lambda := c/R$ (with $R > 0$), $\lambda =: 3 + s^2$, and $\varphi =: \psi a^{1/2}$. Instead of (4.3) we have then

$$\ddot{\psi} + \psi\left[\frac{\Lambda^2}{a^2}(s^2 - 3\alpha) + \frac{\dot{a}^2}{a^2}\left(\frac{1}{4} + 3\alpha\right) - \frac{\ddot{a}}{a}\left(\frac{1}{2} - \alpha\right)\right] = 0 \quad (4.4)$$

If $\alpha = 0$, $B_{\mu\nu}$ conformally scales with the expansion factor, since we obtain as solution of (4.4)

$$\psi_{\pm} = a^{1/2}\exp\left[\mp i\Lambda s\int^{\tau} a^{-1}(\tau)\,d\tau\right] \quad (4.5)$$

for an arbitrary expansion factor $a(\tau)$.

In the static case, $a(\tau) = 1$, we have as solution of (4.4)

$$\psi_{\pm} = \exp[\mp i\Lambda\sqrt{s^2 - 3\alpha}\,\tau] \quad (4.6)$$

In the case of linear expansion, $a(\tau) = \Lambda\tau$, α drops out in equation (4.4), and we obtain

$$\psi_\pm = (\Lambda\tau)^{\mp is+1/2} \tag{4.7}$$

as a pair of fundamental solutions.

Let us now turn to equation (4.2), with $\lambda = 3 + s^2$. The subsidiary conditions (2.12), (2.13), and (2.15) impose restrictions on the solutions, namely

$$\hat{B}^l_l = 0, \qquad \hat{B}_{mn}{}^{\|m} = 0 \tag{4.8}$$

They are consistent with equation (4.2), because

$$B_{mn\|k}{}^{\|k\|n} \equiv B_{mn}{}^{\|n}{}_{\|k}{}^{\|k} + \frac{2}{R^2a^2} B_k{}^k{}_{\|m} - \frac{4}{R^2a^2} B_{mk}{}^{\|k} \tag{4.9}$$

[cf. (2.16) and the formulas for the curvature tensor in the Appendix].

In order to determine the spectral resolution of equation (4.2) under conditions (4.8), we have to specify the spacelike slices. We assume that they are $a(\tau)$-scaled copies of hyperbolic space H^3, as defined at the beginning of the Appendix. The tensorial hyperbolic Laplacian in (4.2) and the divergence in (4.8) are explicitly evaluated in (A.11) and (A.12).

We start with the ansatz

$$\hat{B}_{ij}(s) := \tilde{B}_{ij}(t/R)^{-1-is} \tag{4.10}$$

where $\tilde{B}_{ij}$ is a constant symmetric matrix. In order for $\hat{B}_{ij}(s)$ to satisfy equations (4.2) and (4.8), $\tilde{B}_{ij}$ must be a linear combination of the two matrices

$$\tilde{B}^{(1)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{B}^{(2)} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.11}$$

We generate a complete set of eigenfunctions by applying symmetry transformations of the 3-space metric g_{ij} [defined after (A.9)] to $\hat{B}_{ij}(s)$. It is convenient to use for the H^3 coordinates (x_1, x_2, t) complex notation $(z := x_1 + ix_2, t)$. We consider the Möbius transformation $\alpha_\xi(z) := (z - \xi)^{-1}$ in the complex plane $(\xi = \xi_1 + i\xi_2)$ and lift it to H^3 (Beardon, 1983; Ahlfors, 1981),

$$\alpha_\xi\text{:} \quad (z, t) \to \frac{R^2}{|z - \xi|^2 + t^2} (\overline{z - \xi}, t) \tag{4.12}$$

This coordinate transformation leaves g_{ij} invariant. If we apply it to $\hat{B}_{ij}(s)$, we obtain

$$\tilde{B}_{ij}(t/R)^{-1-is} \to \tilde{B}_{kl}[\alpha'_\xi]^k_i[\alpha'_\xi]^l_j P^{-1-is}(z, t) =: \hat{B}_{ij}(z, t; \xi, s) \tag{4.13}$$

with the Poisson kernel $P(z, t; \xi) := Rt/(|z - \xi|^2 + t^2)$. The Jacobian $[\alpha'_\xi]$ of $\alpha_\xi(z, t)$ explicitly reads

$$[\alpha'_\xi] = \frac{R^2}{(|z - \xi|^2 + t^2)^2}$$

$$\begin{pmatrix} -(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + t^2 & -2(x_1 - \xi_1)(x_2 - \xi_2) & -2(x_1 - \xi_1)t \\ 2(x_1 - \xi_1)(x_2 - \xi_2) & -(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 - t^2 & 2(x_2 - \xi_2)t \\ -2(x_1 - \xi_1)t & -2(x_2 - \xi_2)t & |z - \xi|^2 - t^2 \end{pmatrix} \tag{4.14}$$

In the following we will often use matrix notation, e.g., $\hat{B}(z, t; \xi, s) = [\alpha'_\xi]^t\tilde{B}[\alpha'_\xi]P^{-1-is}(z, t; \xi)$. Because $\alpha_\xi(z, t)$ is a symmetry transformation of the metric, $\hat{B}(z, t; \xi, s)$ is a solution of equations (4.2) and (4.8).

We consider the vector space of all symmetric, complex, three-by-three matrices with scalar product $\langle A, B\rangle := \mathrm{Tr}(A\overline{B})$, and choose an orthonormal basis as

$$\tilde{B}^R = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}^L = \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{B}^{S_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}^{S_2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix} \tag{4.15}$$

$$\tilde{B}^{S_3} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}, \quad \tilde{B}^{S_4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In the following we denote these matrices by $\tilde{B}^X$; X ranges over $(R, L, S_n, n = 1, \ldots, 4)$. $\tilde{B}^R$ and $\tilde{B}^L$ are linear combinations of $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$.

Next we consider the Hilbert space of complex, symmetric (three-by-three) matrix-valued functions on H^3, with the scalar product

$$\langle A, B\rangle_{H^3} := \int_{H^3} a^{-3}\, dV_{H^3}\, \langle A(z, t), B(z, t)\rangle_{H^3}$$

$$\langle A(z, t), B(z, t)\rangle_{H^3} := a^4 g^{ij} g^{kl} A_{ik} \overline{B}_{jl} \tag{4.16}$$

$dV_{H^3} := a^3 R^3 t^{-3}\, dx_1\, dx_2\, dt$ is the volume element of the 3-space.

We define [cf. (4.13)]

$$\hat{B}^X(z, t; \xi, s) := [\alpha'_\xi]^t \tilde{B}^X [\alpha'_\xi] P^{-1-is}(z, t; \xi) \tag{4.17}$$

with $X \in (R, L, S_n, n = 1, \ldots, 4)$, $s \in \mathrm{R}$, $\xi \in \mathrm{C}$. This function system constitutes a complete orthogonal set. In fact, a series of matrix multiplications gives

$$\langle \hat{B}^X(z, t; \xi, s), \hat{B}^Y(z, t; \xi', s')\rangle_{H^3} = P^{1-is}(z, t; \xi)P^{1+is'}(z, t; \xi')\delta^{XY} + O(|\xi - \xi'|) \tag{4.18}$$

from which orthogonality follows [cf. equations (2.10) and (2.13) in Tomaschitz (1993)]. To prove completeness, we only note that the analogue to equation (2.16) in Tomaschitz (1993) is

$$\sum_{X\in(R,L,S_n)} \hat{B}^X_{ij}(z, t; \xi, s)\overline{\hat{B}^X_{kl}(z', t'; \xi, s)}$$
$$= \delta_{ik}\delta_{jl}R^4t^{-2}t'^{-2}P^{1-is}(z, t; \xi)P^{1+is}(z', t'; \xi) + O(|z - z'| + |t - t'|) \tag{4.19}$$

The subspace generated by $\hat{B}^X(z, t; \xi, s)$, $X = R, L$, comprises a complete set of solutions of (4.2) and (4.8). The $\hat{B}^{R,L}$ factorize,

$$\hat{B}^X_{ij}(z, t; \xi, s) = b^X_i(z, t; \xi)b^X_j(z, t; \xi)P^{-1-is}(z, t; \xi) \tag{4.20}$$

with $b^X(z, t; \xi) := \tilde{b}^X[\alpha^i_\xi(z, t)]$, $\tilde{b}^R := (1/\sqrt{2})(1, i, 0)$, and $\tilde{b}^L := (1/\sqrt{2})(i, 1, 0)$. If $\xi = \infty$, then $[\alpha^i_\xi(z, t)]$ is the identity matrix, and $P = t/R$.

Concerning the spin of wave fields, the discussion is quite analogous to that for spin-one-half particles (Tomaschitz, 1994a), and we also sketch that very briefly. We define three contravariant vector fields $\hat{e}_1(t) = (-1, 0, 0)^t t$, $\hat{e}_2(t) = (0, -1, 0)^t t$, and $\hat{e}_3(t) = (0, 0, -1)^t t$ on the horospherical wave fronts issuing at $\xi = \infty$; they are Euclidean planes parallel to the plane at infinity ($t = 0$) of H^3. We write $\hat{e}^k_i$; the subscript i labels the triad vector, the superscript k its components. The spin operators $\hat{\Sigma}_m$ on these wavefronts read (Corson, 1953)

$$\hat{\Sigma}_m{}^i{}_j{}^k{}_l := \delta^i_j\hat{S}_m{}^k{}_l + \delta^k_l\hat{S}_m{}^i{}_j \tag{4.21}$$

The $\hat{S}_m$ are the three spin operators of the electromagnetic field,

$$\hat{S}_m{}^i{}_j := \frac{\hbar}{i}\frac{1}{Ra}\sqrt{\gamma}\epsilon_{mkj}g^{ki}, \qquad \hat{S}_3 = \frac{\hbar}{it}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.22}$$

(The appearance of $\hbar$ is merely symbolic in this classical context.) We have the usual commutator relations $\hat{\Sigma}_i\hat{\Sigma}_j - \hat{\Sigma}_j\hat{\Sigma}_i = i\hbar(1/Ra)\sqrt{\gamma}\epsilon_{ijk}\hat{\Sigma}^k$. We define the spin projections onto the triad vectors as $\hat{S}(i) := \hat{S}_m\hat{e}^m_i$ and $\hat{\Sigma}(i) := \hat{\Sigma}_m\hat{e}^m_i$.

The $b^{R,L}(z, t; \xi)$ as defined after (4.20) represent two circularly polarized states of the electromagnetic field, $\hat{S}(3)b^R = \hbar b^R$, $\hat{S}(3)b^L = -\hbar b^L$, and we have now analogously

$$\hat{\Sigma}(3)\hat{B}^R = 2\hbar\hat{B}^R, \qquad \hat{\Sigma}(3)\hat{B}^L = -2\hbar\hat{B}^L \tag{4.23}$$

Note that the $\hat{\Sigma}(i)\hat{B}^{R,L}$, $i = 1, 2$, are in the orthogonal complement (generated by the $\hat{B}^{S_i}$) of the transverse subspace (generated by $\hat{B}^{R,L}$). The expectation value of $\hat{\Sigma}(3)$, which gives the projection of the spin in (or opposite to) the direction of propagation, is (Bjorken and Drell, 1964)

$$\frac{\langle\alpha\hat{B}^R + \beta\hat{B}^L, \hat{\Sigma}(3)(\alpha\hat{B}^R + \beta\hat{B}^L)\rangle_{H^3}}{|\alpha\hat{B}^R + \beta\hat{B}^L|^2} = 2\hbar\,\frac{\alpha^2|\hat{B}^R|^2 - \beta^2|\hat{B}^L|^2}{\alpha^2|\hat{B}^R|^2 + \beta^2|\hat{B}^L|^2} \tag{4.24}$$

The scalar product is defined in (4.16), $|\hat{B}^X|^2 := \langle\hat{B}^X, \hat{B}^X\rangle_{H^3}$.

Finally we define spin on an arbitrary horosphere $P(z, t; \xi) = \text{const}$. The triad vectors on this horosphere have the components $e_i^k = [\alpha_\xi^{\prime}(z, t)]_l^{-1k}\hat{e}_i^l(\alpha_\xi(z, t))$ [cf. equation (4.2) in Tomaschitz (1994a)]. The horospherical spin operators are $S_{m}{}^{i}{}_{j} := \hat{S}_n{}^k{}_l[\alpha_\xi^{\prime}]^n_m[\alpha_\xi^{\prime}]^{-1i}_k[\alpha_\xi^{\prime}]^l_j$, and Σ_m is as in (4.21) with $\hat{S}_m$ replaced by S_m. The spin in the direction of e_m is defined as $\Sigma(m) = \Sigma_k e_m^k$, and (4.24) holds true with the obvious replacements.

Let us consider a plane wave propagating along the t axis. With $a(\tau) = 1$ and $\psi_\pm$ as in (4.6), we have

$$B_{ij} = \tilde{B}_{ij}(t/R)^{-1} \exp i[-s \log(t/R) \mp \Lambda\sqrt{s^2 - 3\alpha}\,\tau] \tag{4.25}$$

The phase velocity is obtained by equating the differential of the phase to zero, $|v_{\text{ph}}| = a(\tau)Rt^{-1}|dt/d\tau| = c|s|^{-1}(s^2 - 3\alpha)^{1/2}$. To obtain the group velocity, we have to differentiate the phase with respect to s before calculating the differential, $|v_{\text{gr}}| = c|s|(s^2 - 3\alpha)^{-1/2}$. If we agree that gravitational waves propagate with the speed of light, then α must vanish; cf. the end of Section 3. The B field is then conformally coupled [cf. (4.5)] and the energy density is strictly positive [cf. (3.18)]. The Lagrange function (2.6) reads

$$L_G = -\tfrac{1}{4}G_{\kappa\alpha\beta}G^{\kappa\alpha\beta} = \tfrac{1}{2}(E^2 - H^2) \tag{4.26}$$

E^2 and H^2 as introduced in (3.18) scale with $a^{-4}(\tau)$ and so does the energy density.

The general shape of a wave packet is obtained as a superposition of the spectral elementary waves $\varphi_\pm\hat{B}^X$, with $\hat{B}^X$ as in (4.17) and $\varphi_\pm$ as in (4.5) ($\varphi_\pm = a^{1/2}\psi_\pm$). We have

$$B_{ij}^w = \text{Re}\int_{\mathbb{R}^3} ds\, d\xi \sum_{X=R,L} (w_+\varphi_+ + w_-\varphi_-)\hat{B}_{ij}^X(z, t; \xi, s)w(\xi, s, X) \tag{4.27}$$

$w_\pm$ are arbitrary constants and $w(s, \xi, X)$ is a weight function that makes B_{ij}^{w} square-integrable.

5. GRAVITATIONAL WAVES IN A MULTIPLY CONNECTED RW COSMOLOGY

We outline the spectral resolution of the B field for the case that the spacelike slices are multiply connected, open, hyperbolic manifolds [cf. the review by Tomaschitz (1996)]. We assume that the covering group Γ is of Schottky or quasi-Fuchsian type. The spectral theory of equations (4.2) and (4.8) is a straightforward extension of the spectral theory of the electromagnetic field. The formalism and the notation used in this section are explained in Tomaschitz (1993, 1994a).

We study automorphic tensor fields (Poincaré series) of the type

$$B_{ij}^{\Gamma} = \sum_{\gamma\in\Gamma} [\gamma']_i^k B_{kl}(\gamma(z, t))[\gamma']_j^l \tag{5.1}$$

where $B_{kl}(z, t)$ is a symmetric tensor field in H^3. In the following we use matrix notation throughout. $B^{\Gamma}(z, t)$ denotes a tensor field on a hyperbolic 3-manifold (F, Γ). F is a fundamental polyhedron in H^3, and B^{Γ} satisfies periodic boundary conditions on the polyhedral faces; we have $B^{\Gamma}(z, t) = [\beta']^t B(\beta(z, t))[\beta']$ in H^3 for all $\beta \in \Gamma$.

We choose for $B(z, t)$ in (5.1) the $\hat{B}^X(z, t; \xi, s)$ defined in (4.17), and obtain

$$\begin{aligned}&\hat{B}^{X\Gamma}(z, t; \xi, s)\\ &\quad= \sum_{\gamma\in\Gamma} ([\alpha_\xi'(\gamma(z, t))][\gamma'(z, t)])^t \tilde{B}^X[\alpha_\xi'(\gamma(z, t))][\gamma'(z, t)]P^{-1-is}(\gamma(z, t); \xi)\end{aligned} \tag{5.2}$$

which is a solution of equations (4.2) and (4.8), because both α_ξ and the elements of Γ are symmetry transformations of the H^3 metric. With these functions we can construct a complete set of eigenfunctions of (4.2) and (4.8) on the 3-space (F, Γ).

By means of equations (3.6) and (3.7) of Tomaschitz (1993) and equation (7.7) of Tomaschitz (1994a), we can write (5.2) as

$$\begin{aligned}&\hat{B}^{X\Gamma}(z, t; \xi, s)\\ &\quad= \sum_{\gamma\in\Gamma} ([L_\gamma'][\alpha_{\gamma^{-1}\xi}'(z, t)])^t \tilde{B}^X[L_\gamma'][\alpha_{\gamma^{-1}\xi}'(z, t)]\,|\gamma^{-1\prime}\xi|^{-1-is}P^{-1-is}(z, t; \gamma^{-1}\xi)\end{aligned} \tag{5.3}$$

with

$$[L'_\gamma] := \begin{pmatrix} \mathrm{Re}(\gamma^{-1\prime}\xi) & -\mathrm{Im}(\gamma^{-1\prime}\xi) & 0 \\ \mathrm{Im}(\gamma^{-1\prime}\xi) & \mathrm{Re}(\gamma^{-1\prime}\xi) & 0 \\ 0 & 0 & |\gamma^{-1\prime}\xi| \end{pmatrix}$$

$\gamma^{-1\prime}\xi$ denotes the complex derivative of γ^{-1} at ξ. (γ^{-1} acts now in the complex plane as a Möbius transformation.) We have

$$[L'_\gamma]^t \tilde{B}^X [L'_\gamma] = \tilde{B}^X \lambda^X(\gamma^{-1\prime}\xi) \tag{5.4}$$

with complex functions $\lambda^X(z)$ defined by $\lambda^R(z) := \overline{\lambda^L(z)} := z^2$, $\lambda^{S_1}(z) = \lambda^{S_4}(z) = |z|^2$, and $\lambda^{S_2}(z) = \overline{\lambda^{S_3}(z)} = |z|z$. The basis $\tilde{B}^X$ in (4.15) is chosen so that $\lambda^X(z_1z_2) = \lambda^X(z_1)\lambda^X(z_2)$. Equation (5.2) may now be written as

$$\begin{aligned} &\hat{B}^{X\Gamma}(z, t; \xi, s) \\ &= \sum_{\gamma\in\Gamma} [\alpha'_{\gamma^{-1}\xi}(z, t)]^t \tilde{B}^X [\alpha'_{\gamma^{-1}\xi}(z, t)] \lambda^X(\gamma^{-1\prime}\xi) |\gamma^{-1\prime}\xi|^{-1-is} P^{-1-is}(z, t; \gamma^{-1}\xi) \\ &= \sum_{\gamma\in\Gamma} \hat{B}^X(z, t; \gamma^{-1}\xi, s) \lambda^X(\gamma^{-1\prime}\xi) |\gamma^{-1\prime}\xi|^{-1-is} \end{aligned} \tag{5.5}$$

The covering group acts here only on the boundary of H^3 in the complex plane. From (5.5) we easily find

$$\hat{B}^{X\Gamma}(z, t; \xi, s) = \hat{B}^{X\Gamma}(z, t; \beta\xi, s)\lambda^X(\beta'\xi)|\beta'\xi|^{-1-is} \tag{5.6}$$

for all $\beta \in \Gamma$.

It is easy to see that the matrix elements of $\hat{B}^X(z, t; \gamma^{-1}\xi, s)$ are uniformly bounded with respect to the elements of Γ. We write $\lambda = -1 - is$. The matrix elements of $\hat{B}^{X\Gamma}(z, t; \xi, s)$ are then bounded by

$$|\hat{B}^{X\Gamma}_{ij}| < \text{const} \cdot \sum_{\gamma\in\Gamma} |\gamma'\xi|^{2+\lambda} \tag{5.7}$$

This bound is uniform for $\xi \in \cup f_k$ (the f_k denote free faces of F at infinity of H^3) and the series converges for $\mathrm{Re}(2 + \lambda) > \delta$; δ is the Hausdorff dimension of the limit set of Γ. In the case that $\delta > 1$, we define the series (5.2), (5.3), and (5.5) along the abscissa $\mathrm{Re}(\lambda) = -1$ by analytic continuation.

The orthogonality relation is easily verified, and is quite analogous to that for vector fields [cf. equation (3.12) in Tomaschitz (1993)],

$$\langle \hat{B}^{X\Gamma}(s, \xi), \hat{B}^{Y\Gamma}(s', \xi')\rangle_F = 2\pi^3 R^5 s^{-2} \delta^{XY}(\xi - \xi')\delta(s - s') \tag{5.8}$$

Here $\langle\cdot, \cdot\rangle_F$ denotes the Hilbert space scalar product on the 3-space (F, Γ), as in (4.16), but now with the domain of integration H^3 replaced by the polyhedron F.

In the completeness relation there are two square-integrable fields if $\delta > 1$. For $X = S_1, S_4$, we have

$$u^X := \lim_{\lambda \to (\delta - 2)} (\lambda - (\delta - 2)) \sum_{\gamma \in \Gamma} [\alpha'_{\gamma\xi}(z, t)]^t \tilde{B}^X [\alpha'_{\gamma\xi}(z, t)] P^{\delta-2}(z, t; \gamma\xi) |\gamma'\xi|^{2+\lambda} \tag{5.9}$$

We denote the normalized states by $\hat{u}^X$. These states are not eigenfunctions of (4.2); they emerge only if $\delta > 1$, and they are orthogonal to the states generated by the $\hat{B}^{X\Gamma}(\xi, s)$, $X \in (R, L, S_n, n = 1, \ldots, 4)$, $s \in \mathbf{R}$, $\xi \in \cup f_k$. The completeness relation finally reads

$$\sum_{X \in R,L,S_n} \int_{\mathbf{R} \times \cup f_k} d\sigma_{H^3}(s, \xi)\, \hat{B}^{X\Gamma}_{ij}(z, t; \xi, s) \overline{\hat{B}^{X\Gamma}_{kl}(z', t'; \xi, s)} + \hat{u}^{S_1}_{ij}(z, t) \overline{\hat{u}^{S_1}_{kl}(z', t')} + \hat{u}^{S_4}_{ij}(z, t) \overline{\hat{u}^{S_4}_{kl}(z', t')} = g_{ik} g_{jl} \delta_{H^3}(z, t; z', t') \tag{5.10}$$

The spectral parameter ξ ranges only over the free faces f_k of F; the spectral measure $d\sigma_{H^3}$ is given in equation (2.14) of Tomaschitz (1993). [We have put the expansion factor $a(\tau)$, which is irrelevant here, equal to one.] The transverse states are generated by $\hat{B}^{R\Gamma}(\xi, s)$ and $\hat{B}^{L\Gamma}(\xi, s)$. All other states in (5.10) are orthogonal to them, and do not even solve (4.2). $\hat{B}^{R\Gamma}$ and $\hat{B}^{L\Gamma}$ are the circularly polarized states satisfying (4.23). [The spin operators (4.21) and (4.22), together with the tetrad fields on the horospheres, have to be regarded as projected into the 3-manifold by the covering projection.]

With the $\hat{B}^{X\Gamma}$ in (5.5) one can construct wave packets as in (4.27). $\hat{B}^X$ is replaced there by the automorphic fields $\hat{B}^{X\Gamma}$, and the domain of integration of the spectral variable ξ is $\cup f_k$ instead of the whole plane $\mathbf{R}^2$. In practice, however, one proceeds differently [cf. Tomaschitz (1994b)]. One starts with a wave packet (4.27) in the covering space H^3 and projects it onto the 3-manifold by periodization according to (5.1),

$$B^{w\Gamma}_{ij} = \sum_{\gamma \in \Gamma} [\gamma']^k_i B^w_{kl}(\gamma(z, t)) [\gamma']^l_j \tag{5.11}$$

The considerations on energy in Section 3 remain true as they stand, with tensor fields $B^{w\Gamma}_{ij}$ defined on the multiply connected 3-manifold. Geodesic motion in a gravitational wave (5.11) is defined as usual by the perturbed RW metric $g^{B^w\Gamma}_{ij} = g_{ij} + B^{w\Gamma}_{ij}$, $g^{B^w\Gamma}_{00} = -c^2$, $g^{B^w\Gamma}_{0j} = 0$.

6. THE ANALOGUE TO MAXWELL'S EQUATIONS

The Lagrange function (4.26) and the energy-momentum tensor (3.18)–(3.20) are structured as in electromagnetic theory, and so it is very easy to derive the analogue to Maxwell's equations on the spacelike slices of a RW cosmology.

From the potential representation (2.7) we have

$$\frac{1}{\sqrt{-g}}\,\epsilon^{\kappa\rho\mu\nu}G_{\rho\mu\nu} \equiv 0 \tag{6.1}$$

and

$$\frac{1}{\sqrt{-g}}\,\epsilon^{\kappa\mu\nu\sigma}G_{\rho\mu\nu;\sigma} \equiv \frac{1}{\sqrt{-g}}\,\epsilon^{\kappa\mu\nu\sigma}B_{\alpha\nu}R^{\alpha}{}_{\rho\mu\sigma} = 0 \tag{6.2}$$

These identities can be derived by commuting derivatives and using the symmetry properties of the curvature tensor. Here $(-g)^{-1/2}\epsilon^{\kappa\rho\mu\nu}$ is the totally antisymmetric Levi-Civita tensor on the 4-manifold. In a RW geometry, with $B_{\mu\nu}$ fields satisfying the subsidiary conditions (2.12), (2.13), and (2.15), the right side of (6.2) vanishes.

The wave equation (2.10) reads, under the given conditions (namely $\alpha = \beta = 0$, subsidiary conditions, and curvature tensor of a RW geometry),

$$G_{\rho\gamma\nu}{}^{;\gamma} = 0 \tag{6.3}$$

This wave equation and the identities (3.10), (6.1), and (6.2) are analogous to the manifestly covariant Maxwell equations $F_{\mu\nu}{}^{;\nu} = 0$ and $(-g)^{-1/2}\epsilon^{\mu\nu\kappa\lambda}F_{\nu\kappa;\lambda} = 0$. The subsidiary conditions (2.12), (2.13), and (2.15) correspond to the Lorentz condition $A_{\mu}{}^{;\mu} = 0$ and the Coulomb gauge $A_0 = 0$.

To obtain the analogue of Maxwell's equations on the 3-slices, we first have to express the 4-dimensional covariant derivatives $G_{\alpha\beta\gamma;\delta}$ in (6.2), (6.3), and (3.10) by covariant differentiation ($\|$) on the 3-space. This is done in (A.7) and (A.8). Then we simply insert E and H via (3.15) and (3.17). From the defining equations and the subsidiary conditions it is easy to see that E_{ij} and H_{ij} are symmetric and have vanishing trace, $E^n_n = H^n_n = 0$.

From (6.2) ($\kappa = 0$, $\rho = l$) we obtain

$$H^{lm}{}_{\|m} = 0 \tag{6.4}$$

and from (3.10) ($\rho = 0$, $\nu = n$)

$$E_{nl}{}^{\|l} = 0 \tag{6.5}$$

From (6.2) ($\kappa = k$, $\rho = l$) we have

$$\frac{1}{c}\frac{1}{a^2}\,(a^2H^k_l)_{,0} - \gamma^{-1/2}\epsilon^{kmj}E_{lm\|j} = 0 \tag{6.6}$$

and from the wave equation (6.3) ($\rho = l$, $\nu = 0$) we finally obtain

$$\gamma^{1/2}\epsilon_{nkm}H^{k\|m}_l + \frac{1}{c}\,E_{ln,0} = 0 \tag{6.7}$$

Equations (6.4)–(6.7) are the gravitational analogue to the vacuum Maxwell equations in RW cosmology.

7. CONCLUDING REMARKS

We have constructed here a wave mechanics of linear gravitational waves on RW background geometries. This wave mechanics is self-consistent as a linear theory, and it admits a straightforward definition of a positive energy density for wave packets. This is achieved by making extensive use of the electromagnetic formalism.

The theory developed is not meant as a linearized theory of gravity. There are no source terms in the evolution equation (6.3), which is designed for gravitational waves freely propagating on the RW background. In background metrics of lesser symmetry the electromagnetic formalism would break down, because the subsidiary (gauge) conditions (2.12), (2.13), and (2.15) become inconsistent with the wave equation. As in electrodynamics there remains some gauge freedom in the wave equation, even with the three gauge conditions imposed. We can easily find wave fields which satisfy $G_{\rho\mu\nu} = 0$ as well as the gauge conditions. In the case of vanishing curvature (e.g., a Minkowski universe or RW cosmology with linear expansion factor and negatively curved 3-space) we may simply choose $B_{\mu\nu} = \xi_{;\mu;\nu}$. Here ξ is a scalar independent of cosmic time which satisfies the Laplace–Beltrami equation on the 3-space. However, such solutions of the wave equation do not correspond to gravitational fields. We defined the most general shape of a gravitational wave packet in equations (4.27) and (5.11). These wave packets propagate with the speed of light. The weight function in (4.27) must be chosen so that B_{ij}^{w} and its derivatives are square-integrable with respect to the volume element of the 3-space. [If $w(s, \xi, X)$ is Gaussian with respect to both spectral variables s and ξ, this certainly works out, in (4.27) as well as in (5.11).] Then the wave pulse has a well-defined energy that scales with the inverse of the expansion factor, $a(\tau)E = \text{const}$.

In the wave equation (2.9) we have ultimately chosen $\alpha = \beta = 0$ because of the three conditions summarized in the Introduction. In the short-wave approximation, Misner *et al.* (1973) put $\alpha = -1$, $\beta = 1$; Isaacson (1968) chooses $\alpha = \beta = 1$ in this approximation scheme; Landau and Lifshitz (1962) choose $\alpha = \beta = -1$. The linear wave equation is then only approximately consistent with the three subsidiary conditions and/or the requirement that the gravitational pulse propagates with the speed of light. But for all these choices of α and β the authors come to the same conclusion, namely that (4.27) is the generic shape of a linearized wave packet; cf. also the discussion at the end of Section 4.

In a RW cosmology linear gravitational waves exactly satisfy a wave equation which is consistent with the gauge conditions imposed. The wave equation is conformally coupled and makes it possible to treat linear gravitational waves in close analogy to electrodynamics. We have demonstrated this here by means of Maxwell's equations, the energy-momentum tensor, and the spin of wave fields.

I should finally mention that my initial motivation to design a self-consistent linear formalism originated in the study of gravitational waves in RW geometries with multiply connected spacelike slices, as outlined in Section 5. The 'method of images' as indicated in (5.1) would give a fairly uncontrollable result unless the periodized wave field is an exact solution of the wave equation.

APPENDIX. THE CURVATURE TENSOR IN RW COSMOLOGIES AND SOME EXPLICIT FORMULAS FOR COVARIANT DIFFERENTIATION

The RW metric $g_{\mu\nu}$ is defined as $g_{00} = -c^2$, $g_{ij} = a^2(\tau)\tilde{g}_{ij}$, and $g_{0j} = 0$, where $\tilde{g}_{ij}$ is a metric of constant curvature $-1/R^2$ on the 3-space (R may be real, imaginary, or ∞). $a(\tau)$ is the expansion factor. We denote the determinant of g_{ij} by γ, Latin indices run from 1 to 3, and Greek indices run from 0 to 3. For $g_{\mu\nu}$ we have the Christoffel symbols

$$\Gamma^n_{0m} = \delta^n_m \frac{\dot{a}}{a}, \qquad \Gamma^0_{kl} = g_{kl} \frac{1}{c^2} \frac{\dot{a}}{a} \tag{A.1}$$

The symbols with two and three zero-indices vanish in a RW geometry, and the symbols with spatial indices Γ^i_{jk} are time independent. Many calculations of this paper are performed without specifying the sign of the curvature of $\tilde{g}_{ij}$ and without a special coordinate representation of the 3-space. Only in Sections 4 and 5 do we assume that the 3-space has negative curvature ($R > 0$), and we use there as coordinate representation the Poincaré half-space H^3, with rectangular coordinates (x_1, x_2, t), $t > 0$, and $\tilde{g}_{ij} = R^2 t^{-2} \delta_{ij}$ (cf. Tomaschitz, 1993, 1994a,b).

The nonzero components of the Riemann tensor [with sign conventions as in Landau and Lifshitz (1962) and Misner *et al.* (1973)] are

$$R_{klmn} = \left(\frac{1}{R^2 a^2} - \frac{1}{c^2}\frac{\dot{a}^2}{a^2}\right)(g_{kn}g_{lm} - g_{km}g_{ln}), \qquad R_{0l0n} = -g_{ln}\frac{\ddot{a}}{a} \tag{A.2}$$

All other nonvanishing components can be obtained by using the symmetry with respect to the interchange of the first and second index pair and the skew-symmetry within these index pairs.

The nonvanishing components of the first covariant derivatives of the Riemann tensor are

$$R_{klmn;0} = -2\frac{\dot a}{a}\left(\frac{1}{R^2a^2} - \frac{1}{c^2}\frac{\dot a^2}{a^2} + \frac{1}{c^2}\frac{\ddot a}{a}\right)(g_{kn}g_{lm} - g_{km}g_{ln})$$

$$R_{0lmn;i} = -\frac{\dot a}{a}\left(\frac{1}{R^2a^2} - \frac{1}{c^2}\frac{\dot a^2}{a^2} + \frac{1}{c^2}\frac{\ddot a}{a}\right)(g_{in}g_{lm} - g_{im}g_{ln})$$

$$R_{0l0n;0} = -\frac{d}{d\tau}\left(\frac{\ddot a}{a}\right)g_{ln} \tag{A.3}$$

The nonvanishing components of the Ricci tensor read

$$R_{ln} = g_{ln}\left(\frac{-2}{R^2a^2} + \frac{1}{c^2}\frac{\ddot a}{a} + 2\frac{1}{c^2}\frac{\dot a^2}{a^2}\right), \qquad R_{00} = -3\frac{\ddot a}{a} \tag{A.4}$$

Its derivatives are

$$R_{0n;l} = 2g_{ln}\frac{\dot a}{a}\left(\frac{1}{R^2a^2} + \frac{1}{c^2}\frac{\ddot a}{a} - \frac{1}{c^2}\frac{\dot a^2}{a^2}\right), \qquad R_{00;0} = -3\frac{d}{d\tau}\frac{\ddot a}{a}$$

$$R_{mn;0} = g_{mn}\left[4\frac{\dot a}{a}\left(\frac{1}{R^2a^2} + \frac{1}{c^2}\frac{\ddot a}{a} - \frac{1}{c^2}\frac{\dot a^2}{a^2}\right) + \frac{1}{c^2}\frac{d}{d\tau}\frac{\ddot a}{a}\right] \tag{A.5}$$

All other components vanish or can be obtained by the symmetry in the first and second indices.

Next we give some formulas which relate four-dimensional covariant differentiation (RW metric $g_{\mu\nu}$) with covariant differentiation on the spacelike slices (metric g_{ij}). In a RW geometry this is comparatively simple, because $\Gamma^{(3)i}{}_{jk} = \Gamma^{(4)i}{}_{jk}$, i.e., the Christoffel symbols $\Gamma^{(3)}$ of the 3-metric g_{ij} coincide with the Christoffel symbols $\Gamma^{(4)}$ (with spatial indices) of $g_{\mu\nu}$. Therefore we can drop the superscripts (3), (4). We denote three-dimensional covariant differentiation on the 3-slices by a double stroke (‖). A subscript comma followed by zero denotes ordinary differentiation with respect to cosmic time τ. If $B_{\mu\nu}$ is a symmetric tensor field on the 4-manifold, then B_{mn} is a symmetric tensor field on the 3-slices, B_{0m} is a 3-vector, and B_{00} is a scalar on the 3-space of the RW cosmology (τ is then regarded as a parameter labeling the 3-slices, and we consider coordinate transformations on a given 3-slice). We have

$$B_{mn;l} = B_{mn\|l} - \frac{1}{c^2}\frac{\dot a}{a}g_{lm}B_{n0} - \frac{1}{c^2}\frac{\dot a}{a}g_{ln}B_{m0}, \qquad B_{mn;0} = B_{mn,0} - 2\frac{\dot a}{a}B_{mn}$$

$$B_{m0;l} = B_{m0\|l} - \frac{\dot a}{a}B_{lm} - \frac{1}{c^2}\frac{\dot a}{a}g_{lm}B_{00}, \qquad B_{m0;0} = B_{m0,0} - \frac{\dot a}{a}B_{m0}$$

$$B_{00;l} = B_{00\|l} - 2\frac{\dot a}{a}B_{l0}, \qquad B_{00;0} = B_{00,0} \tag{A.6}$$

All other components can be obtained by the symmetry in the first two indices.

For the G field defined in (2.7) we have as nonvanishing components

$$G_{lmn} = B_{ln\|m} - B_{lm\|n} + \frac{1}{c^2}\frac{\dot{a}}{a} g_{ln} B_{0m} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{lm} B_{0n}$$

$$G_{lm0} = B_{l0\|m} - B_{lm,0} + \frac{\dot{a}}{a} B_{lm} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{lm} B_{00}$$

$$G_{0mn} = B_{0n\|m} - B_{0m\|n}, \qquad G_{0m0} = B_{00\|m} - B_{0m,0} - \frac{\dot{a}}{a} B_{0m} \tag{A.7}$$

G is of course skew in the last two indices. For its derivatives we have

$$G_{lmn;k} = G_{lmn\|k} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{kl} G_{0mn} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{km} G_{l0n} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{kn} G_{lm0}$$

$$G_{0mn;k} = G_{0mn\|k} - \frac{\dot{a}}{a} G_{kmn} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{km} G_{00n} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{kn} G_{0m0}$$

$$G_{lmn;0} = G_{lmn,0} - 3\frac{\dot{a}}{a} G_{lmn}, \qquad G_{0mn;0} = G_{0mn,0} - 2\frac{\dot{a}}{a} G_{0mn}$$

$$G_{l0n;k} = G_{l0n\|k} - \frac{\dot{a}}{a} G_{lkn} - \frac{1}{c^2}\frac{\dot{a}}{a} g_{kl} G_{00n}, \qquad G_{l0n;0} = G_{l0n,0} - 2\frac{\dot{a}}{a} G_{l0n}$$

$$G_{00n;k} = G_{00n\|k} - \frac{\dot{a}}{a} G_{k0n} - \frac{\dot{a}}{a} G_{0kn}, \qquad G_{00n;0} = G_{00n,0} - \frac{\dot{a}}{a} G_{00n} \tag{A.8}$$

All other components are zero or can be obtained from the antisymmetry in the second and third indices.

To perform the time separation in the wave equation (cf. Section 4) one has to express the tensorial d'Alembertian $B_{\mu\nu;\alpha}{}^{;\alpha}$ by covariant derivatives on the 3-slices,

$$\begin{aligned} B_{mn;\alpha}{}^{;\alpha} = {} & B_{mn\|k}{}^{\|k} - \frac{1}{c^2} B_{mn,0,0} + \frac{1}{c^2}\frac{\dot{a}}{a} B_{mn,0} - 2\frac{1}{c^2}\frac{\dot{a}}{a}(B_{0n\|m} + B_{0m\|n}) \\ & + 2\frac{1}{c^2}\frac{\dot{a}^2}{a^2} B_{mn} + 2\frac{1}{c^2}\frac{\ddot{a}}{a} B_{mn} + 2\frac{1}{c^4}\frac{\dot{a}^2}{a^2} g_{mn} B_{00} \end{aligned}$$

$$\begin{aligned} B_{0m;\alpha}{}^{;\alpha} = {} & B_{0m\|k}{}^{\|k} - \frac{1}{c^2} B_{0m,0,0} - 2\frac{\dot{a}}{a} B_{mk}{}^{\|k} - \frac{1}{c^2}\frac{\dot{a}}{a} B_{0m,0} + \frac{1}{c^2}\frac{\ddot{a}}{a} B_{0m} \\ & - 2\frac{1}{c^2}\frac{\dot{a}}{a} B_{00,m} + 7\frac{1}{c^2}\frac{\dot{a}^2}{a^2} B_{0m} \end{aligned}$$

$$B_{00;\alpha}{}^{;\alpha} = B_{00\|k}{}^{\|k} - \frac{1}{c^2} B_{00,0,0} - 3\frac{1}{c^2}\frac{\dot{a}}{a} B_{00,0} - 4\frac{\dot{a}}{a} B_{0k}{}^{\|k} + 2\frac{\dot{a}^2}{a^2} B_k{}^k + 6\frac{1}{c^2}\frac{\dot{a}^2}{a^2} B_{00} \tag{A.9}$$

Here $B_{mn\|k}{}^{\|k}$, $B_{0m\|k}{}^{\|k}$, and $B_{00\|k}{}^{\|k}$ are the tensorial, vectorial, and scalar Laplacians on the 3-slices. We now evaluate these Laplacians for the case that the 3-slices are $a(\tau)$-scaled copies of hyperbolic space H^3 [metric $g_{ij} = a^2(\tau)R^2t^{-2}\delta_{ij}$]. The Christoffel indices are

$$\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^3_{33} = -\Gamma^3_{11} = -\Gamma^3_{22} = -t^{-1} \tag{A.10}$$

all other three-indices are zero or obtained by interchanging the lower indices. We obtain

$$\begin{aligned}
a^2R^2B_{mn\|k}{}^{\|k} &= t^2\Delta_E B_{mn} - 2B_{mn} + 2B_{33}\delta_{mn} + 3tB_{mn,3} - 2t(B_{3n,m} + B_{3m,n})\\
a^2R^2B_{3n\|k}{}^{\|k} &= t^2\Delta_E B_{3n} - 5B_{3n} - 2tB_{33,n} + 3tB_{3n,3} + 2t(B_{1n,1} + B_{2n,2})\\
a^2R^2B_{33\|k}{}^{\|k} &= t^2\Delta_E B_{33} + 3tB_{33,3} - 4B_{33} + 2(B_{11} + B_{22}) + 4t(B_{13,1} + B_{23,2})\\
a^2R^2B_{0n\|k}{}^{\|k} &= t^2\Delta_E B_{0n} - 2B_{0n} + tB_{0n,3} - 2tB_{30,n}\\
a^2R^2B_{03\|k}{}^{\|k} &= t^2\Delta_E B_{03} - 3B_{03} + t(2B_{01,1} + 2B_{02,2} + B_{03,3})\\
a^2R^2B_{00\|k}{}^{\|k} &= t^2\Delta_E B_{00} - tB_{00,3}
\end{aligned} \tag{A.11}$$

The subscript comma indicates as always ordinary derivatives, and the subscripts 1, 2, 3 denote differentiation with respect to x_1, x_2, and t, respectively. The indices n, m run in (A.11) only over 1, 2. Here $\Delta_E := \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_t$ is the scalar Euclidean Laplacian in the half-space H^3.

The components of the 3-divergence in (4.8) read

$$\begin{aligned}
a^2R^2B_{mk}{}^{\|k} &= t^2(B_{m1,1} + B_{m2,2} + B_{m3,3}) - tB_{m3}\\
a^2R^2B_{3k}{}^{\|k} &= t^2(B_{31,1} + B_{32,2} + B_{33,3}) + t(B_{11} + B_{22})
\end{aligned} \tag{A.12}$$

m runs here again only over 1, 2.

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