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# High-index asymptotics of spherical Bessel products averaged with modulated Gaussian power laws

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**Abstract.** Bessel integrals of type $\int_0^\infty g(k)j_l^{(m)}(k)j_l^{(n)}(k)k^2\mathrm{d}k$ are investigated, where the kernel $g(k)$ is a modulated Gaussian power-law distribution $k^\mu \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$, and the $j_l^{(m)}$ are multiple derivatives of spherical Bessel functions. These integrals define the multipole moments of Gaussian random fields on the unit sphere, arising in multipole fits of temperature and polarization power spectra of the cosmic microwave background. Two methods allowing efficient numerical calculation of these integrals are presented, covering Bessel indices $l$ in the currently accessible multipole range $0 \le l \le 10^4$ and beyond. The first method is based on a representation of spherical Bessel functions by Lommel polynomials. Gaussian power-law averages can then be calculated in closed form as finite Hankel series of parabolic cylinder functions, which allow high-precision evaluation. The second method is asymptotic, covering the high-$l$ regime, and is applicable to general distribution functions $g(k)$ in the integrand; it is based on the uniform Nicholson approximation of the Bessel derivatives in conjunction with an integral representation of squared Airy functions. A numerical comparison of these two methods is performed, employing Gaussian power laws and Kummer distributions to average the Bessel products.

**Mathematics Subject Classification (2000).** 33C10, 33F05.

**Keywords.** Spherical Bessel functions, Airy functions, High-index asymptotics, Nicholson approximation, Lommel polynomials, Hankel series, Gaussian power-law densities, Kummer distributions.

## 1. Introduction

We study a class of Bessel integrals,

$$D_{\mathrm{ssB}}^{(m,n)}(l,p,\mu;a,b,\omega) = \int_0^\infty k^{\mu+2}\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k} j_l^{(m)}(pk) j_l^{(n)}(pk)\mathrm{d}k, \tag{1.1}$$

arising in the multipole expansion of isotropic Gaussian random fields on the unit sphere. In the integrand, $j_l(x)$ is a spherical Bessel function of integer index $l \ge 0$, the superscripts $(m)$ and $(n)$ denote multiple derivatives thereof, $p$ is a positive scale parameter, $a > 0, b$ and $\omega$ are real constants, and we assume a real power-law exponent $\mu > -3$. The integrals are then safely convergent for the complete set of spherical Bessel functions and their derivatives. Occasionally, we will also consider other parameter ranges, such as $\mathrm{Re}\, a > 0$ or $\mathrm{Re}\, a = 0$ with $b > 0$, or analytic continuation in $\mu$.

Recent observations of temperature fluctuations in the cosmic microwave background (CMB) radiation could resolve multipole moments $C_l$ up to $l \sim 10^4$, and the spherical Green function of the CMB temperature power spectrum was reconstructed from a spectral fit of data sets covering the complete multipole range accessible today, cf. [1] and references therein. The $C_l$ coefficients of the isotropic temperature correlation function are assembled from Bessel integrals of type (1.1). A multipole fit of the currently observable CMB temperature power spectrum thus requires to calculate these integrals with Bessel indices ranging in the interval $0 \le l \le 10^4$. Here, we develop two independent and practically viable methods to do this. The first makes use of the fact that spherical Bessel functions admit a finite representation in terms of Lommel polynomials and trigonometric functions. The second method is asymptotic, based on a uniform high-$l$ Airy approximation of the Bessel derivatives $j_l^{(m)}$ and an integral representation of squared Airy functions.

In Sect. 2, we define the spherical Bessel functions used in the subsequent sections. Spherical Bessel functions $j_l(x)$ are effectively elementary functions, a sine and a cosine each multiplied with a polynomial in $1/x$. In Sect. 3 and Appendix A, we discuss the polynomial representation of squared spherical Bessel (ssB) functions, $j_l^2(x)$ being defined by three polynomials in $1/x$ whose coefficients are obtained as finite series of products of Hankel symbols. In Sect. 4, we discuss Lommel polynomials, in particular Lommel's identity, and how they are related to squared spherical Bessel functions.

In Sect. 5 and Appendix B, we explain the Hankel decomposition of integrals of type

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g) = \int_0^\infty g(k) j_l^{(m)}(pk) j_l^{(n)}(pk) k^2 \mathrm{d}k. \tag{1.2}$$

We first perform the decomposition with a general kernel function $g(k)$ and then specify a modulated Gaussian power-law distribution, $g(k) = k^\mu \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$, as in (1.1). The method used to calculate the integrals (1.2) makes use of the fact that the Bessel product in the integrand is an elementary function, composed of trigonometric functions and high-degree polynomials in $1/k$. We use term-by-term integration and analytic continuation in the power-law index $\mu$ to reduce the integrals to a finite series of confluent hypergeometric functions. At low and intermediate Bessel index, up to about $l \sim 300$, this method is quite efficient and suitable for high-precision calculations. At higher index, the coefficients of the polynomials defining the squared spherical Bessel functions become too large, and it is preferable to employ asymptotic methods.

In Sect. 6 and Appendix C, we use a uniform Airy approximation of spherical Bessel functions paired with an integral representation of squared Airy functions to obtain a numerically tractable high-$l$ limit of the integrals (1.2). This can be done for a general kernel function $g(k)$. The high-$l$ asymptotics of (1.2) turns out to be very sensitive as to whether the two Bessel derivatives in the integrand are both even/odd or mixed. (The Bessel index $l$ is an arbitrary integer $l \geq 0$, the same for both derivatives.) In Sect. 7, we specify the kernel function as a modulated Gaussian, deriving explicit formulas for the high-$l$ asymptotics of the integrals (1.1) suitable for numerical evaluation.

In Sect. 8, we present our conclusions. In Tables 1, 2, 3, and Figs. 1, 2, we perform a numerical comparison of the asymptotic Airy approximation of integral (1.1) with a high-precision calculation based on the finite Hankel expansion discussed in Sect. 5.2. We do this comparison over a wide range of Bessel indices, testing integrands with squared Bessel functions $j_l^2$ and $j_l'^2$ as well as the mixed product $j_l j_l'$. In Table 1, we use a real kernel function, a Gaussian power-law density $g(k) = k^\mu \mathrm{e}^{-ak^2-bk}$, cf. (1.2), and in Tables 2 and 3 a Kummer distribution $g(k) = k^\mu \mathrm{e}^{-(b+\mathrm{i}\omega)k}$ (power law with modulated exponential cutoff). The exponents $(\mu, a, b, \omega)$ defining these distributions are taken from the multipole fit of the CMB temperature power spectrum in [1], extending over five multipole decades.

## 2. Polynomial representation of spherical Bessel functions

Spherical Bessel functions are related to ordinary Bessel functions $J_\nu(x)$ by the renormalization [2,3]

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \tag{2.1}$$

where $n$ is a non-negative integer. We start with the Hankel representation [4,5]

$$J_\nu(x) = \frac{1}{2}(H_\nu^{(1)}(x) + H_\nu^{(2)}(x)), \quad H_\nu^{(2)}(x) = H_\nu^{(1)*}(x), \tag{2.2}$$

TABLE 1. *Hankel expansion and Airy approximation of the integrals* $D_{\rm ssB}^{(m,n)}(l)=\int_0^\infty k^{\mu+2}e^{-ak^2-(b+{\rm i}\omega)k}j_l^{(m)}(pk)j_l^{(n)}(pk){\rm d}k$, *cf. (7.11)*

| Bessel index $l$ | $D_{\rm ssB}^{(0,0)}(l)$ Hankel series (5.21) & (5.10) | $D_{\rm ssB}^{(0,0)}(l)$ Airy approx. (7.10) & (7.3) | $D_{\rm ssB}^{(1,1)}(l)$ Hankel series (B.9) & (5.21) | $D_{\rm ssB}^{(1,1)}(l)$ Airy approx. (7.10) & (7.3) | $D_{\rm ssB}^{(0,1)}(l)$ Hankel series (B.10) & (5.21) | $D_{\rm ssB}^{(0,1)}(l)$ Airy approx. (7.9) & (7.3) |
|---|---|---|---|---|---|---|
| 0 | 532.938174613 | 532.944 | 532.997589023 | 532.533 | −6.3069291017 | −6.36494 |
| 1 | 532.997589023 | 533.003 | 531.870974224 | 531.679 | −5.8074247067 | −5.81611 |
| 5 | 533.636652778 | 533.640 | 527.842081688 | 527.760 | −5.1700427817 | −5.17074 |
| 10 | 535.141547899 | 535.144 | 521.984426143 | 521.927 | −4.8550188303 | −4.85520 |
| 20 | 539.906507889 | 539.908 | 507.765561401 | 507.725 | −4.5358616717 | −4.53589 |
| 30 | 546.370891555 | 546.372 | 490.744741343 | 490.713 | −4.3428618793 | −4.34285 |
| 50 | 562.013437328 | 562.012 | 450.079272067 | 450.059 | −4.0419443961 | −4.04190 |
| 100 | 590.094818491 | 590.089 | 325.613074617 | 325.615 | −3.0043589257 | −3.00433 |
| 150 | 549.492171445 | 549.488 | 198.974283889 | 198.989 | −1.3271020964 | −1.32716 |
| 200 | 418.200922119 | 418.205 | 100.147172638 | 100.160 | 0.26031552915 | 0.260235 |
| 300 | 111.387440579 | 111.390 | 12.9233638834 | 12.9194 | 0.78759203123 | 0.787644 |
| 400 | 9.32585024713 | 9.32400 | 0.60624953554 | 0.60440 | 0.12672210047 | 0.126718 |
| 500 | 0.23230119755 | 0.23204 | 0.00950364817 | 0.00940 | 0.00466018262 | 0.004656 |
| $10^3$ | – | $1.8278\times10^{-17}$ | – | $1.6841\times10^{-19}$ | – | $9.4917\times10^{-19}$ |

The superscripts $(m)$ and $(n)$ of the spherical Bessel functions denote derivatives, so that $D_{\rm ssB}^{(0,0)}(l)$ refers to the square $j_l^2$ in the integrand, $D_{\rm ssB}^{(1,1)}(l)$ to the squared first derivative $j'^2_l$, and $D_{\rm ssB}^{(0,1)}(l)$ to the product $j_lj'_l$, cf. (B.1)–(B.3). The Hankel expansion is explained in Sect. 5 and Appendix B, the Airy approximation in Sects. 6 and 7 and Appendix C. We calculate the integrals $D_{\rm ssB}^{(m,n)}(l)$ with the parameter values $p=1$, cf. (5.22), $\mu=0$, cf. after (5.20), $a=6.26\times10^{-5}, b=-0.02$, and $\omega=0$. (These parameters are taken from a multipole fit of the CMB temperature power spectrum [1].) In the first column, we list the order $l$ of the spherical Bessel functions. In the second column, we list the integrals $D_{\rm ssB}^{(0,0)}(l)=D_{\rm ssB}$ assembled via (5.21) with the finite Hankel series defined in (5.10), (5.18) and (5.20). In the third column, we list the same integral $D_{\rm ssB}^{(0,0)}(l)$ calculated in the high-$l$ Airy approximation (7.10) (with $I^{(0,0)}(\alpha;\mu)$ in parametrization (7.3) and (7.4)). In the 4th and 6th column, we list the integrals $D_{\rm ssB}^{(1,1)}(l)$ and $D_{\rm ssB}^{(0,1)}(l)$, respectively, which have been reduced to linear combinations of integrals of type $D_{\rm ssB}^{(0,0)}$ according to (B.9) and (B.10). The latter are evaluated as described above, by means of (5.21) and the Hankel series in (5.10), whose coefficients are calculated via (5.18) and (5.20). The high-$l$ approximations of the integrals $D_{\rm ssB}^{(1,1)}(l)$ and $D_{\rm ssB}^{(0,1)}(l)$ are listed in the 5th and 7th column, calculated by way of the asymptotic formulas stated in (7.10) and (7.9), respectively (with the integrals $I^{(m,n)}$ defined in (7.3) and (7.4) substituted). The indicated 12 digits of the Hankel evaluation are all significant, the numbers being truncated, whereas the Airy approximation is rounded to six digits. A pictorial overview of the accuracy of the Airy approximation as a function of Bessel index is given in Figs. 1 and 2

where $\nu$ is a positive index, $x$ a real argument, and the star indicates complex conjugation. The Hankel functions are elementary for positive half-integer index $\nu=n+1/2, n\ge0$,

$$H_{n+1/2}^{(1)}(x)=\sqrt{\frac{2}{\pi x}}\frac{{\rm e}^{{\rm i}x}}{{\rm i}^{n+1}}\sum_{m=0}^{n}\frac{{\rm i}^m}{2^m m!}\frac{\Gamma(n+1+m)}{\Gamma(n+1-m)}\frac{1}{x^m}. \tag{2.3}$$

We write the spherical Bessel functions (2.1) in exponential notation,

$$j_n(x)=\frac{1}{2{\rm i}^{n+1}}\frac{{\rm e}^{{\rm i}x}}{x}D_n(x)+{\rm c.c.},\quad D_n(x)=\sum_{k=0}^{n}\frac{d_k(n)}{x^k}, \tag{2.4}$$

where c.c. means ‘complex conjugated,’ and the coefficients of polynomial $D_n(x)$ read

$$d_k(n)=\frac{{\rm i}^k}{2^k k!}\frac{\Gamma(n+1+k)}{\Gamma(n+1-k)}=\frac{{\rm i}^k}{2^k}[n,k], \tag{2.5}$$

Table 2. *Comparison of Hankel expansion and Airy approximation of the integrals* $D_{\mathrm{ssB}}^{(m,n)}(l)$ *in (7.11)*

| Bessel index $l$ | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(0,0)}(l)$ Hankel series (5.21) & (5.10) | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(0,0)}(l)$ Airy approx. (7.10) & (7.3) | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(1,1)}(l)$ Hankel series (B.9) & (5.21) | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(1,1)}(l)$ Airy approx. (7.10) & (7.3) | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(0,1)}(l)$ Hankel series (B.10) & (5.21) | $\mathrm{Re}\,D_{\mathrm{ssB}}^{(0,1)}(l)$ Airy approx. (7.9) & (7.3) |
|---|---|---|---|---|---|---|
| 0 | −943.64449766 | −943.512 | −942.13942814 | −944.090 | −4.7567154046 | −4.75141 |
| 1 | −942.13942814 | −942.067 | −944.64610787 | −946.035 | −4.7101791551 | −4.70544 |
| 5 | −930.78142485 | −930.794 | −963.71459411 | −964.368 | −4.1497753261 | −4.14594 |
| 10 | −914.83799237 | −914.898 | −1000.5143551 | −1000.78 | −2.7167680005 | −2.71371 |
| 20 | −909.13219435 | −909.239 | −1088.5945685 | −1088.44 | 2.11364333876 | 2.115110 |
| 30 | −969.08680355 | −969.204 | −1166.6530192 | −1166.27 | 8.77231508434 | 8.772094 |
| 50 | −1341.0233325 | −1341.07 | −1212.4745145 | −1211.94 | 23.6305743822 | 23.62741 |
| 100 | −2934.8735740 | −2934.58 | −462.90617429 | −463.040 | 31.9755290847 | 31.97320 |
| 150 | −2093.7144922 | −2093.63 | 755.577704904 | 754.8922 | −17.309316498 | −17.3035 |
| 200 | 2279.93569941 | 2279.416 | 1098.26718765 | 1098.123 | −55.051552477 | −55.0467 |
| 300 | 3122.58326388 | 3122.991 | −547.89562663 | −547.274 | 41.0038470081 | 40.99436 |
| 400 | −5474.5797847 | −5474.34 | −183.91545245 | −184.645 | 9.76194311307 | 9.772742 |
| 500 | 2405.59163224 | 2404.667 | 418.127221889 | 418.4321 | −45.843233956 | −45.8481 |
| $10^3$ | −1313.7707375 | −1314.06 | 0.49358255048 | 0.326624 | −4.4914266232 | −4.48624 |
| $10^4$ | – | $-2.113\times10^{-14}$ | – | $-1.940\times10^{-16}$ | – | $3.7483\times10^{-16}$ |

The caption of Table 1 applies. The integrals $D_{\mathrm{ssB}}^{(m,n)}(l)$ have a complex integrand defined by the parameter set $p=1, \mu=1, a=0, b=4.6\times10^{-3}$, and $\omega=2.15\times10^{-2}$ [1]. We list the real part of $D_{\mathrm{ssB}}^{(m,n)}(l)$, that is, the integrals $\mathrm{Re}\,D_{\mathrm{ssB}}^{(m,n)}(l)=\int_0^\infty k^{\mu+2}\mathrm{e}^{-ak^2-bk}\cos(\omega k)j_l^{(m)}(pk)j_l^{(n)}(pk)\mathrm{d}k$. As for the Hankel expansion, the coefficients of the Hankel series (5.10) are calculated via (5.19), without invoking the confluent functions in (5.18), since the quadratic term in the exponential of the integrand vanishes ($a=0$). Otherwise, the Hankel expansion is performed as outlined in the caption to Table 1, by making use of (B.9), (B.10), (5.21), and (5.10). The Airy approximations are based on (7.9) and (7.10). Typically, 3-digit precision is reached by the Airy approximation; even at very low Bessel index $l$, the first two digits are safe. In the case of $\mathrm{Re}\,D_{\mathrm{ssB}}^{(1,1)}(l=10^3)$, the Airy approximation is noticeably worse; the real part of integral $D_{\mathrm{ssB}}^{(1,1)}(l=10^3)$ is by two orders smaller than its imaginary part, cf. Table 3

so that $d_k^*(n)=(-1)^k d_k(n)$. Here, $[n,k]$ is Hankel's symbol, usually denoted by $(1/2+n,k)$ [4],

$$[n,k]=\frac{1}{\Gamma(1+k)}\frac{\Gamma(n+1+k)}{\Gamma(n+1-k)}=\frac{(n+k)!}{k!(n-k)!}, \tag{2.6}$$

which satisfies the identities

$$\begin{aligned}[n,k]&=\frac{1}{k!}(1+n-k)_{2k}=\frac{1}{k!}(1+n)_k(1+n-k)_k,\\ [n+j,k]&=[n,k]\frac{(n+1+k)_j}{(n+1-k)_j},\end{aligned} \tag{2.7}$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)=a(a+1)\cdots(a+k-1)$ is the rising factorial (Pochhammer symbol).

We split the polynomial $D_n(x)$ in (2.4) into a real and imaginary part, $D_n(x)=A_n(x)+\mathrm{i}B_n(x)$, and write $j_n(x)$ in (2.4) as

$$j_n(x)=\frac{\sin x}{x}E_n(x)+\frac{\cos x}{x}F_n(x), \tag{2.8}$$

where

$$\begin{aligned}E_n(x)&=A_n(x)\cos\frac{\pi n}{2}+B_n(x)\sin\frac{\pi n}{2},\\ F_n(x)&=B_n(x)\cos\frac{\pi n}{2}-A_n(x)\sin\frac{\pi n}{2}.\end{aligned} \tag{2.9}$$

Table 3. *Imaginary part of the integrals* $D_{\rm ssB}^{(m,n)}(l)$ *in (7.11), calculated via Hankel expansion and Airy approximation*

| Bessel index $l$ | Im $D_{\rm ssB}^{(0,0)}(l)$ Hankel series (5.21) & (5.10) | Im $D_{\rm ssB}^{(0,0)}(l)$ Airy approx. (7.10) & (7.3) | Im $D_{\rm ssB}^{(1,1)}(l)$ Hankel series (B.9) & (5.21) | Im $D_{\rm ssB}^{(1,1)}(l)$ Airy approx. (7.10) & (7.3) | Im $D_{\rm ssB}^{(0,1)}(l)$ Hankel series (B.10) & (5.21) | Im $D_{\rm ssB}^{(0,1)}(l)$ Airy approx. (7.9) & (7.3) |
|---|---|---|---|---|---|---|
| 0 | −423.21889637 | −423.304 | −423.89883250 | −423.134 | 22.2432277307 | 22.25018 |
| 1 | −423.89883250 | −423.983 | −423.21745379 | −422.454 | 22.3178185229 | 22.32217 |
| 5 | −433.32184326 | −433.399 | −413.71578397 | −412.971 | 22.9466526900 | 22.94751 |
| 10 | −459.40293850 | −459.464 | −386.87393752 | −386.173 | 24.0487981155 | 24.04781 |
| 20 | −550.47181451 | −550.483 | −286.49982957 | −285.940 | 26.0375272436 | 26.03475 |
| 30 | −670.78827046 | −670.737 | −132.35086706 | −131.981 | 26.5645108625 | 26.56112 |
| 50 | −861.55131999 | −861.389 | 281.906579574 | 281.8465 | 20.7450292734 | 20.74265 |
| 100 | 303.872169450 | 303.9029 | 1203.53587476 | 1202.937 | −24.219717752 | −24.2150 |
| 150 | 3566.28115293 | 3565.860 | 962.852648538 | 962.9013 | −45.297502311 | −45.2943 |
| 200 | 4420.19116366 | 4419.963 | −161.89771028 | −161.168 | 0.37347105835 | 0.367247 |
| 300 | −4743.2343515 | −4742.53 | −649.03793654 | −649.517 | 44.7920101496 | 44.79558 |
| 400 | −197.09430723 | −198.039 | 588.065999753 | 587.9305 | −57.410504942 | −57.4066 |
| 500 | 4166.99835104 | 4167.473 | −115.03218245 | −114.460 | 22.7415256490 | 22.73070 |
| $10^3$ | 320.100736931 | 319.6052 | 61.1351285649 | 61.21254 | −13.820398389 | −13.8240 |
| $10^4$ | – | $-3.910\times10^{-14}$ | – | $5.6328\times10^{-17}$ | – | $-3.112\times10^{-16}$ |

The captions to Tables 1 and 2 apply. The integrals are calculated with the parameter values $p=1,\mu=1,a=0,b=4.6\times10^{-3}$, and $\omega=2.15\times10^{-2}$ as in Table 2. Here, we study $\mathrm{Im}\,D_{\rm ssB}^{(m,n)}(l)=-\int_0^\infty k^{\mu+2}\mathrm{e}^{-ak^2-bk}\sin(\omega k)j_l^{(m)}(pk)j_l^{(n)}(pk)\mathrm{d}k$. The Hankel expansion of these integrals is based on (B.9), (B.10) and (5.21), compiled with the finite Hankel series defined in (5.10), (5.20) and (5.19). The Hankel expansion is compared with the Airy approximation of the respective integrals based on the asymptotic high-$l$ limits (7.9) and (7.10). The indicated digits of the Hankel series evaluation are significant, and usually 3-digit precision is reached by the Airy approximation

In particular, $j_0(x)=\sin x/x$. More explicitly, by making use of $A_n(x)=(D_n+D_n^*)/2$ and $B_n(x)=(D_n-D_n^*)/(2\mathrm{i})$ as well as (2.5), we find

$$E_n(x)=\sum_{k=0}^{n}\cos\frac{\pi(k-n)}{2}\frac{1}{2^k k!}\frac{\Gamma(n+1+k)}{\Gamma(n+1-k)}\frac{1}{x^k},$$
$$F_n(x)=\sum_{k=0}^{n}\sin\frac{\pi(k-n)}{2}\frac{1}{2^k k!}\frac{\Gamma(n+1+k)}{\Gamma(n+1-k)}\frac{1}{x^k}. \tag{2.10}$$

In the Hankel representation (2.4), we use

$$D_n(x)=A_n(x)+\mathrm{i}B_n(x),$$
$$A_n(x)=\sum_{m=0}^{[n/2]}\frac{d_{2m}(n)}{x^{2m}},\quad \mathrm{i}B_n(x)=\sum_{m=0}^{[(n-1)/2]}\frac{d_{2m+1}(n)}{x^{2m+1}}, \tag{2.11}$$

where the brackets of the upper summation boundaries indicate the largest integer less than or equal to the argument (floor function), and the coefficients $d_k(n)$ are defined in (2.5). In this way, we obtain explicit formulas for the polynomials $A_n(x)$ and $B_n(x)$,

$$A_n(x)=\sum_{m=0}^{[n/2]}\frac{(-1)^m}{2^{2m}(2m)!}\frac{\Gamma(n+1+2m)}{\Gamma(n+1-2m)}\frac{1}{x^{2m}},$$
$$B_n(x)=\sum_{m=0}^{[(n-1)/2]}\frac{(-1)^m}{2^{2m+1}(2m+1)!}\frac{\Gamma(n+2+2m)}{\Gamma(n-2m)}\frac{1}{x^{2m+1}}. \tag{2.12}$$

If the upper summation boundary is negative, the series $B_n(x)$ is void, $B_0(x)=0$.

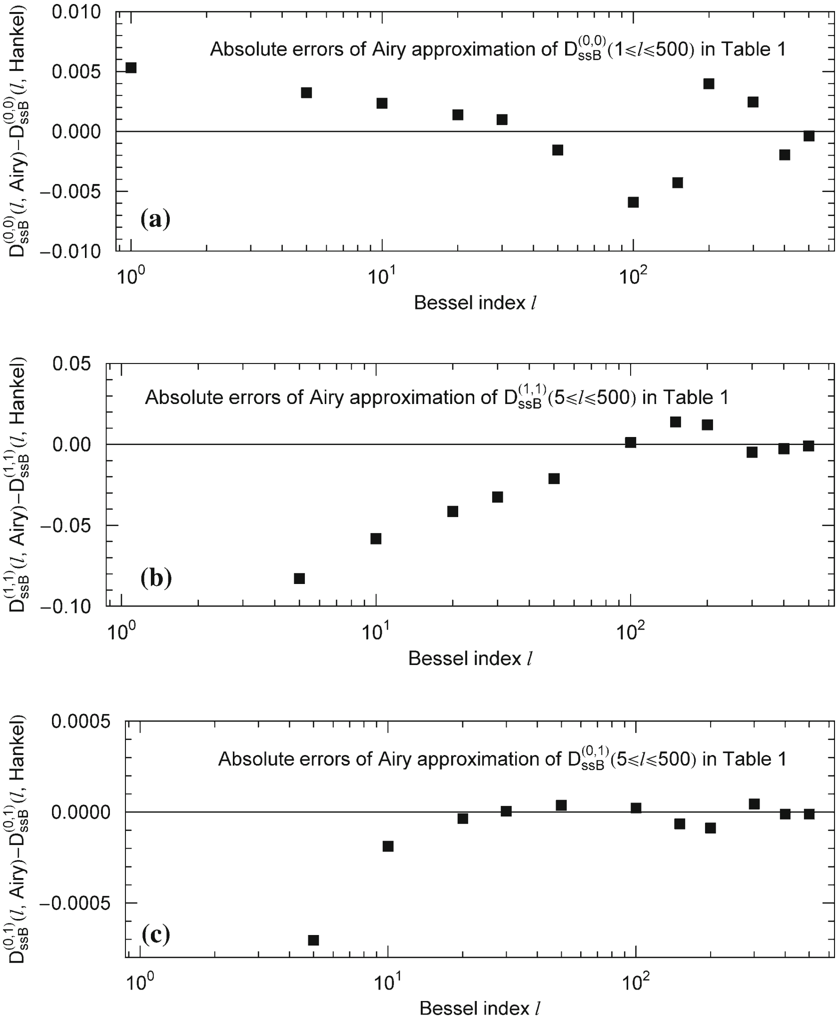


FIG. 1. **a**–**c** Absolute errors $\delta D_{\text{ssB}}^{(m,n)}(l) = D_{\text{ssB}}^{(m,n)}(l, \text{Airy}) - D_{\text{ssB}}^{(m,n)}(l, \text{Hankel})$ of the Airy approximation, compared with the exact Hankel series evaluation of the integrals recorded in Table 1. The absolute errors at first decrease with increasing Bessel index $l$ and then start to fluctuate. The errors of the $l = 0$ and $l = 1$ integrals in Table 1 lie outside the depicted coordinate range: $\delta D_{\text{ssB}}^{(0,0)}(0) \approx 5.8 \times 10^{-3}$ (Fig. 1a), $\delta D_{\text{ssB}}^{(1,1)}(0) \approx -0.46$ and $\delta D_{\text{ssB}}^{(1,1)}(1) \approx -0.19$ (Fig. 1b), $\delta D_{\text{ssB}}^{(0,1)}(0) \approx -5.8 \times 10^{-2}$ and $\delta D_{\text{ssB}}^{(0,1)}(1) \approx -8.7 \times 10^{-3}$ (Fig. 1c). The ordinate scales of the figures substantially differ, depending on the Bessel derivatives $(m, n)$ in the integrand, see the caption to Table 1

## 3. Squared spherical Bessel functions

We square $j_n(x)$ in Hankel representation (2.4),

$$j_n^2(x) = (-1)^{n+1} \frac{\mathrm{e}^{2\mathrm{i}x}}{4x^2} D_n^2(x) + \text{c.c.} + \frac{1}{2x^2} D_n(x) D_n^*(x), \tag{3.1}$$

where the polynomial $D_n(x)$ is defined in (2.4) and (2.5), and split the squared polynomial $D_n^2$ as

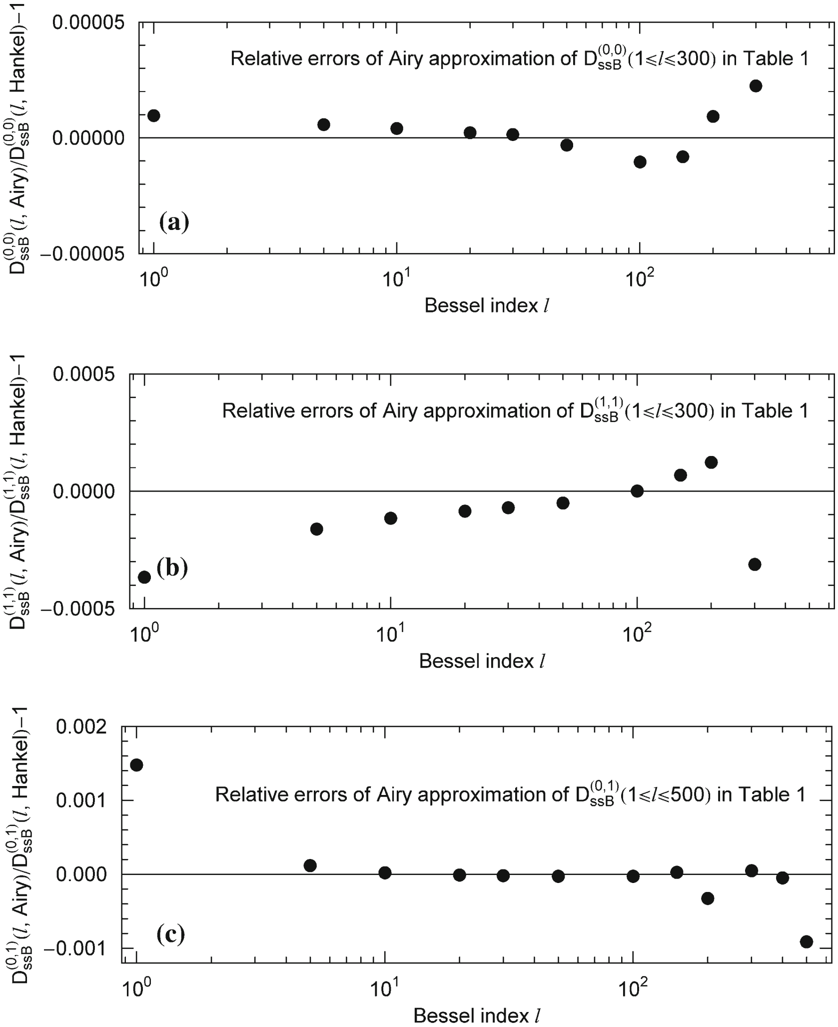


FIG. 2. **a**–**c** Relative errors $\Delta D_{\mathrm{ssB}}^{(m,n)}(l) = D_{\mathrm{ssB}}^{(m,n)}(l, \mathrm{Airy})/D_{\mathrm{ssB}}^{(m,n)}(l, \mathrm{Hankel}) - 1$ of the Airy approximation of the integrals in Table 1. The errors of the $l = 0$ integrals lie outside the coordinate range: $\Delta D_{\mathrm{ssB}}^{(0,0)}(0) \approx 1.1 \times 10^{-5}$ (Fig. 2a), $\Delta D_{\mathrm{ssB}}^{(1,1)}(0) \approx -8.7 \times 10^{-4}$ (Fig. 2b), and $\Delta D_{\mathrm{ssB}}^{(0,1)}(0) \approx 9.2 \times 10^{-3}$ (Fig. 2c). The $l = 400$ and $l = 500$ errors are also outside the ordinate range of Fig. 2a, b: $\Delta D_{\mathrm{ssB}}^{(0,0)}(400) \approx -2.0 \times 10^{-4}, \Delta D_{\mathrm{ssB}}^{(0,0)}(500) \approx -1.1 \times 10^{-3}$, and $\Delta D_{\mathrm{ssB}}^{(1,1)}(400) \approx -3.1 \times 10^{-3}, \Delta D_{\mathrm{ssB}}^{(1,1)}(500) \approx -1.1 \times 10^{-2}$. Like the absolute error, the relative error at first decreases with increasing Bessel index $l$. At high $l$, it starts to increase again, when the integrals become very small as compared with integrals depending on low and intermediate Bessel indices, cf. Table 1

$$D_n^2(x) = \sum_{k=0}^{2n} \frac{a_k(n)}{x^k} = \hat{A}_n(x) + \mathrm{i}\hat{B}_n(x),$$
$$\hat{A}_n(x) = \sum_{k=0}^{n} \frac{a_{2k}(n)}{x^{2k}}, \quad \hat{B}_n(x) = \frac{1}{\mathrm{i}} \sum_{k=0}^{n-1} \frac{a_{2k+1}(n)}{x^{2k+1}} = \sum_{k=0}^{n-1} \frac{b_{2k+1}(n)}{x^{2k+1}}. \tag{3.2}$$

The coefficients $a_{2k}(n)$ and $b_{2k+1}(n) = -\mathrm{i}a_{2k+1}(n)$ are real, and $\hat{B}_0(x) = 0$. The squared absolute value of $D_n(x)$ can be written as

$$\hat{C}_n(x) = D_n(x)D_n^*(x) = \sum_{k=0}^{n} \frac{c_{2k}(n)}{x^{2k}}. \tag{3.3}$$

There are no odd terms in series (3.3), since $d_k^*(n) = (-1)^k d_k(n)$ in (2.4). If we refrain from complex notation and use real harmonics in (3.1), we find

$$j_n^2(x) = \frac{1}{2x^2}\left((-1)^{n+1}\hat{A}_n(x)\cos 2x + (-1)^n \hat{B}_n(x)\sin 2x + \hat{C}_n(x)\right). \tag{3.4}$$

The polynomial coefficients $a_k(n)$ and $c_{2k}(n)$ in (3.2) and (3.3) can be obtained from the coefficients $d_k(n)$ [defining $j_n(x)$ in (2.4) and (2.5)] by means of the product formulas (A.9) and (A.10). The coefficients of polynomial $\hat{A}_n(x)$ read

$$\begin{aligned} a_{2k\le n}(n) &= d_k^2(n) + 2\sum_{m=0}^{k-1} d_m(n)d_{2k-m}(n),\\ a_{2k\ge n}(n) &= -d_k^2(n) + 2\sum_{m=0}^{n-k} d_{2k-n+m}(n)d_{n-m}(n), \end{aligned} \tag{3.5}$$

where $0 \le k \le n$. The coefficients of polynomial $\hat{B}_n(x)$ in (3.2) are

$$\begin{aligned} a_{2k+1\le n}(n) &= 2\sum_{m=0}^{k} d_m(n)d_{2k+1-m}(n),\\ a_{2k+1\ge n}(n) &= 2\sum_{m=0}^{n-k-1} d_{2k+1-n+m}(n)d_{n-m}(n), \end{aligned} \tag{3.6}$$

where $0 \le k \le n-1$. The coefficients of the squared absolute value $\hat{C}_n(x)$ in (3.3) read

$$\begin{aligned} c_{2k\le n}(n) &= (-1)^k d_k^2(n) + 2\sum_{m=0}^{k-1} (-1)^m d_m(n)d_{2k-m}(n),\\ c_{2k\ge n}(n) &= (-1)^{k+1} d_k^2(n) + 2\sum_{m=0}^{n-k} (-1)^{n-m} d_{2k-n+m}(n)d_{n-m}(n), \end{aligned} \tag{3.7}$$

where $0 \le k \le n$. In these formulas, we substitute $d_k(n) = (\mathrm{i}/2)^k[n,k]$, cf. (2.5), to obtain the coefficients as linear combinations of products of Hankel symbols.

First, cf. (3.5),

$$\begin{aligned} a_{2k\le n}(n) &= \frac{(-1)^k}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{k-1}[n,m][n,2k-m]\\ &= -\frac{(-1)^k}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{k}[n,m][n,2k-m]. \end{aligned} \tag{3.8}$$

If the upper summation boundary is negative, the sum is void. Alternatively, we can replace the upper summation boundary $k-1$ by $k$ if we change the sign of the first term as done in the second identity in (3.8). (The coefficients $a_{2k}(n)$ in (3.5) and $c_{2k}(n)$ in (3.7) are defined for integer $k$ in the range $0 \le k \le n, n \ge 0$.)

As for the coefficients $a_{2k\ge n}(n)$ in (3.5), we find

$$a_{2k\geq n}(n) = -\frac{(-1)^k}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{n-k}[n,2k-n+m][n,n-m]$$
$$= \frac{(-1)^k}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{n-k-1}[n,2k-n+m][n,n-m]. \quad (3.9)$$

We can replace the upper summation boundary $n-k$ by $n-k-1$ if we drop the minus sign in front of the first term, as done in the second identity. In this case, we use the convention that the sum is void if the upper summation boundary $n-k-1$ is negative, which happens for $k=n$.

Regarding the odd coefficients in (3.6), we obtain

$$a_{2k+1\leq n}(n) = \mathrm{i}b_{2k+1\leq n}(n) = \mathrm{i}\frac{(-1)^k}{2^{2k}}\sum_{m=0}^{k}[n,m][n,2k+1-m] \quad (3.10)$$

and

$$a_{2k+1\geq n}(n) = \mathrm{i}b_{2k+1\geq n}(n) = \mathrm{i}\frac{(-1)^k}{2^{2k}}\sum_{m=0}^{n-k-1}[n,2k+1-n+m][n,n-m]. \quad (3.11)$$

The coefficients $a_{2k+1}(n)$ are defined for integer $k$ in the range $0 \leq k \leq n-1, n \geq 1$.

As for the $c_{2k\leq n}(n)$ in (3.7), we find

$$c_{2k\leq n}(n) = \frac{1}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{k-1}(-1)^m[n,m][n,2k-m]$$
$$= -\frac{1}{2^{2k}}[n,k]^2 + \frac{(-1)^k}{2^{2k-1}}\sum_{m=0}^{k}(-1)^m[n,m][n,2k-m], \quad (3.12)$$

and the remark following (3.8) applies. Finally, the $c_{2k\geq n}(n)$ in (3.7) read

$$c_{2k\geq n}(n) = -\frac{1}{2^{2k}}[n,k]^2 + \frac{(-1)^{k+n}}{2^{2k-1}}\sum_{m=0}^{n-k}(-1)^m[n,2k-n+m][n,n-m]$$
$$= \frac{1}{2^{2k}}[n,k]^2 + \frac{(-1)^{k+n}}{2^{2k-1}}\sum_{m=0}^{n-k-1}(-1)^m[n,2k-n+m][n,n-m], \quad (3.13)$$

and the remark after (3.9) applies.

The coefficients $c_{2k}(n)$ in (3.12) and (3.13) defining the polynomial $\hat{C}_n(x)$ in (3.3) can be summed by way of Lommel's identity, cf. Sect. 4:

$$c_{2k}(n) = \frac{[n,k]}{2^{2k}}\frac{\Gamma(2k+1)}{\Gamma(k+1)}, \quad [n,k] = \frac{1}{\Gamma(1+k)}\frac{\Gamma(n+1+k)}{\Gamma(n+1-k)}, \quad (3.14)$$

valid for $n \geq 0$. The coefficients $a_k(n)$ defining the polynomials $\hat{A}_n(x)$ and $\hat{B}_n(x)$ in (3.2) cannot be summed in closed form and have to be calculated by means of the series (3.8)–(3.11).

We list the first two coefficients of the polynomials $\hat{A}_n(x), \hat{B}_n(x)$, and $\hat{C}_n(x)$ in (3.2) and (3.3),

$$a_0(n) = 1, \quad a_2(n) = -\frac{1}{2}n(n+1)(n^2+n-1),$$
$$a_1(n) = \mathrm{i}n(n+1), \quad a_3(n) = -\frac{\mathrm{i}}{24}(n+2)(n+1)n(n-1)(4n^2+4n-6),$$
$$c_0(n) = 1, \quad c_2(n) = \frac{1}{2}n(n+1). \quad (3.15)$$

The highest coefficient is obtained by putting $k = n$ or $k = n - 1$ in the preceding series,

$$a_{2n}(n) = \frac{(-1)^n}{2^{2n}} \frac{\Gamma^2(2n+1)}{\Gamma^2(n+1)} = \frac{(-1)^n}{\pi} 2^{2n} \Gamma^2(1/2+n),$$
$$a_{2n-1}(n) = -2\mathrm{i} a_{2n}(n), \quad c_{2n}(n) = (-1)^n a_{2n}(n). \tag{3.16}$$

The leading-order asymptotics of the gamma function,

$$\Gamma(n-\mu) = \sqrt{2\pi} \frac{\mathrm{e}^{n(\log n - 1)}}{n^{1/2+\mu}} \left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right), \tag{3.17}$$

valid for $n \to \infty$ and fixed real $\mu$, indicates that the series coefficients become very large even for moderate degree $n$. In fact, the representation of the Hankel functions in (2.3) can be recovered from an asymptotic series for $H_\nu^{(1)}(x)$ valid for fixed real index $\nu$ and large argument [6]. At half-integer index $\nu = n + 1/2$, this series terminates after a finite number of terms, degenerating into the exact formula (2.3), which can be used for small argument (large ratios $1/x^m$ in (2.3)) as well, irrespectively of the magnitude of the series coefficients. Similarly, the spherical Bessel functions in representation (2.4) or (2.8) and their squares in (3.1) and (3.4) are elementary functions, and the exact identity (2.8) can be obtained from an asymptotic expansion of $J_\nu(x)$ with fixed real index [4] which becomes a finite series at half-integer $\nu$, cf. (2.1), (2.8), and (2.10).

It is instructive to compare these Hankel representations to the ascending series expansions [5]

$$j_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2^n \Gamma(n+k+1)}{\Gamma(2n+2k+2)} x^{2k+n}, \tag{3.18}$$

$$j_n^2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2^{2n+2k} \Gamma^2(n+k+1) x^{2k+2n}}{\Gamma(2n+k+2)\Gamma(2n+2k+2)}, \tag{3.19}$$

which can be obtained via the finite series (2.8) and (3.4) by substituting the ascending series of the respective trigonometric functions. The singular $1/x$ powers of the polynomials in (2.8) and (3.4) all cancel, as well as the regular powers up to the leading orders of (3.18) and (3.19). In effect, high-precision arithmetics is required if we use the finite representation (3.4) of $j_n^2(x)$ for high-order $n$ and small argument $x$, since the three polynomial terms constituting (3.4) become very large and nearly cancel each other out, also see the discussion following (5.21).

## 4. Lommel polynomials

The spherical Bessel functions (2.8) can be written in terms of Lommel polynomials,

$$j_n(x) = \frac{\sin x}{x} R_{n,1/2}(x) - \frac{\cos x}{x} R_{n-1,3/2}(x), \tag{4.1}$$

where $n$ is a non-negative integer, and $R_{n,1/2+k}(x)$ is the polynomial [4,5]

$$R_{n,1/2+k}(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{2^{2m-n} x^{2m-n}}{\Gamma(n-2m+1)} \frac{\Gamma(m+k+1)}{\Gamma(m+1)}$$
$$\times \frac{\Gamma(n-m+1)}{\Gamma(n-m+k+1)} \frac{\Gamma(2n-2m+2k+1)}{\Gamma(2m+2k+1)}. \tag{4.2}$$

The index $k$ can be complex, cf. the analytic continuation discussed after (4.14). The brackets of the upper summation boundary denote the floor function. In particular,

$$R_{n,1/2}(x) = \sum_{m=0}^{[n/2]} (-1)^m 2^{2m-n} x^{2m-n} [n, n-2m],$$
$$R_{n-1,3/2}(x) = \sum_{m=0}^{[(n-1)/2]} (-1)^m 2^{2m-n+1} x^{2m-n+1} [n, n-2m-1], \tag{4.3}$$

where $[n, k]$ is Hankel's symbol defined in (2.6).

We may identify, for arbitrary non-negative integer $n$, cf. (2.8)–(2.10),

$$E_n(x) = R_{n,1/2}(x), \quad F_n(x) = -R_{n-1,3/2}(x), \tag{4.4}$$

so that $E_0 = 1$ and $F_0 = 0$. For $n$ even, cf. (2.9),

$$E_n(x) = (-1)^{n/2} A_n(x) = R_{n,1/2}(x),$$
$$F_n(x) = (-1)^{n/2} B_n(x) = -R_{n-1,3/2}(x), \tag{4.5}$$

where, cf. (4.3),

$$R_{n,1/2}(x) = \sum_{j=0}^{n/2} (-1)^{n/2-j} \frac{[n, 2j]}{2^{2j}} \frac{1}{x^{2j}},$$
$$R_{n-1,3/2}(x) = \sum_{j=0}^{n/2-1} (-1)^{n/2-1-j} \frac{[n, 2j+1]}{2^{2j+1}} \frac{1}{x^{2j+1}}. \tag{4.6}$$

The identities (4.6) are valid for even $n$, and $R_{-1,3/2} = 0$. For odd integer $n$, we find, analogously to (4.5) and (4.6),

$$E_n(x) = (-1)^{(n-1)/2} B_n(x) = R_{n,1/2}(x),$$
$$F_n(x) = (-1)^{(n+1)/2} A_n(x) = -R_{n-1,3/2}(x), \tag{4.7}$$

with the Lommel polynomials (odd $n$)

$$R_{n,1/2}(x) = \sum_{j=0}^{(n-1)/2} (-1)^{(n-1)/2-j} \frac{[n, 2j+1]}{2^{2j+1}} \frac{1}{x^{2j+1}},$$
$$R_{n-1,3/2}(x) = \sum_{j=0}^{(n-1)/2} (-1)^{(n-1)/2-j} \frac{[n, 2j]}{2^{2j}} \frac{1}{x^{2j}}. \tag{4.8}$$

On squaring (2.8), we obtain, for non-negative integer $n$,

$$j_n^2(x) = \frac{1}{x^2} \left( E_n^2(x) \sin^2 x + F_n^2(x) \cos^2 x + 2E_n(x) F_n(x) \sin x \cos x \right). \tag{4.9}$$

Alternatively, we may substitute $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$ into (3.4),

$$j_n^2(x) = \frac{1}{x^2} \left( \hat{E}_n(x) \sin^2 x + \hat{F}_n(x) \cos^2 x + (-1)^n \hat{B}_n(x) \sin x \cos x \right), \tag{4.10}$$

where we defined the polynomials, cf. (3.2) and (3.3),

$$\hat{E}_n(x) = \frac{1}{2} \left( \hat{C}_n(x) + (-1)^n \hat{A}_n(x) \right), \quad \hat{F}_n(x) = \frac{1}{2} \left( \hat{C}_n(x) - (-1)^n \hat{A}_n(x) \right). \tag{4.11}$$

Accordingly, we can identify

$$E_n^2(x) = \hat{E}_n(x), \quad F_n^2(x) = \hat{F}_n(x), \quad 2E_n(x)F_n(x) = (-1)^n \hat{B}_n(x), \tag{4.12}$$

so that, cf. (4.4),

$$\hat{E}_n(x) = R^2_{n,1/2}(x), \quad \hat{F}_n(x) = R^2_{n-1,3/2}(x). \tag{4.13}$$

We note Lommel's identity [5]

$$R^2_{n,1/2}(x) + R^2_{n-1,3/2}(x) = (-1)^n R_{2n,1/2-n}(x), \tag{4.14}$$

where $R_{2n,1/2-n}(x)$ is to be understood as the $\varepsilon \to 0$ limit of the polynomial $R_{2n,1/2-n+\varepsilon}(x)$, cf. (4.2). We thus find, cf. (4.11),

$$\hat{C}_n(x) = \hat{E}_n(x) + \hat{F}_n(x) = (-1)^n R_{2n,1/2-n+\varepsilon}(x), \tag{4.15}$$

where $\varepsilon \to 0$, and $\hat{C}_n(x)$ is the polynomial defined in (3.3), (3.12), and (3.13).

It remains to evaluate $R_{2n,1/2-n+\varepsilon}(x)$ in the limit $\varepsilon \to 0$, for arbitrary non-negative integer $n$. To this end, we start with series $R_{n,1/2+k}(x)$ in (4.2), replace $n \to 2n$, and put $k = -n + \varepsilon$. In the series coefficients, we employ the $\varepsilon$ expansion

$$\frac{\Gamma(-j+\varepsilon)}{\Gamma(-2j+2\varepsilon)} = (-1)^j \frac{2\Gamma(2j+1)}{\Gamma(j+1)} + \mathrm{O}(\varepsilon), \tag{4.16}$$

valid for non-negative integer $j$, and obtain

$$R_{2n,1/2-n+\varepsilon}(x) = (-1)^n \sum_{j=0}^{n} 2^{-2j} x^{-2j} \frac{\Gamma(n+1+j)\Gamma(2j+1)}{\Gamma(n+1-j)\Gamma^2(j+1)} + \mathrm{O}(\varepsilon), \tag{4.17}$$

to be substituted into (4.15). In the limit $\varepsilon \to 0$, we can thus identify the coefficients of polynomial $\hat{C}_n(x)$ in (3.3) as

$$c_{2j}(n) = 2^{-2j} \frac{\Gamma(n+1+j)\Gamma(2j+1)}{\Gamma(n+1-j)\Gamma^2(j+1)} = \frac{[n,j]}{2^{2j}} \frac{\Gamma(2j+1)}{\Gamma(j+1)}, \tag{4.18}$$

where $[n,j]$ is Hankel's symbol (2.6). In this way, we have summed the finite series (3.12) and (3.13).

## 5. Averaging squared spherical Bessel functions with modulated Gaussian power laws

### 5.1. Hankel decomposition

We consider the Bessel integrals

$$D_{\mathrm{ssB}}(l, p, \mu; f) = \int_0^\infty k^{\mu+2} f(k) j_l^2(pk) \mathrm{d}k, \tag{5.1}$$

where $f(k)$ is an arbitrary complex function, $p$ a positive scale parameter, and $\mu$ a real exponent. Occasionally, we will perform analytic continuation in $\mu$. At this point, it is not necessary to specify the kernel function $f(k)$, or to discuss convergence properties, but we will later study $f(k) = \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$, with real exponents $a > 0, b$ and $\omega$, or $a = 0$ and positive $b$, cf. Sect. 5.2. The factor $k^2$ in (5.1) stems from a volume integration in polar coordinates [1], and $\omega$ is the modulation frequency of the Gaussian power-law distribution $k^\mu \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$ over which the squared spherical Bessel function $j_l^2(pk)$ is averaged.

We start with the unilateral Fourier transform

$$D_{\exp}(p, \mu; f) = \int_0^\infty k^\mu f(k) \exp(2\mathrm{i}pk) \mathrm{d}k, \tag{5.2}$$

and split the exponential $\exp(2\mathrm{i}pk)$ into real and imaginary parts, defining the trigonometric functionals

$$D_{\cos}(p,\mu;f) = \int_0^\infty k^\mu f(k)\cos(2pk)\mathrm{d}k$$
$$= \frac{1}{2}(D_{\exp}(p,\mu;f) + D_{\exp}(-p,\mu;f)), \tag{5.3}$$
$$D_{\sin}(p,\mu;f) = \int_0^\infty k^\mu f(k)\sin(2pk)\mathrm{d}k$$
$$= \frac{1}{2\mathrm{i}}(D_{\exp}(p,\mu;f) - D_{\exp}(-p,\mu;f)) \tag{5.4}$$

and

$$D_0(\mu;f) = \int_0^\infty k^\mu f(k)\mathrm{d}k = D_{\exp}(p=0,\mu;f). \tag{5.5}$$

The functionals $D_{\exp}$ and $D_{\cos,\sin,0}$ can be analytic continuations in $\mu$ of the respective integrals.

In the Bessel integral (5.1), we substitute the Hankel representation (3.4) of the squared spherical Bessel function,

$$j_l^2(x) = \frac{1}{2x^2}\left((-1)^{l+1}\hat{A}_l(x)\cos 2x + (-1)^l\hat{B}_l(x)\sin 2x + \hat{C}_l(x)\right), \tag{5.6}$$

where $\hat{A}_l(x), \hat{B}_l(x)$, and $\hat{C}_n(x)$ are polynomials defined in (3.2) and (3.3):

$$\hat{A}_l(x) = \sum_{k=0}^{l}\frac{a_{2k}(l)}{x^{2k}}, \quad \hat{B}_l(x) = \sum_{k=0}^{l-1}\frac{b_{2k+1}(l)}{x^{2k+1}}, \quad \hat{C}_l(x) = \sum_{k=0}^{l}\frac{c_{2k}(l)}{x^{2k}}, \tag{5.7}$$

and $\hat{B}_0(x) = 0$. The polynomial coefficients $a_{2k}(l), b_{2k+1}(l)$ and $c_{2k}(l)$ are listed in (3.8)–(3.11) and (3.14). In this way, we can split the Bessel integral (5.1) into three components, each defined by a polynomial in (5.7),

$$D_{\mathrm{ssB}}(l,p,\mu;f) = \int_0^\infty k^{\mu+2}f(k)j_l^2(pk)\mathrm{d}k$$
$$= \frac{1}{2p^2}\left((-1)^{l+1}D_{\mathrm{ssB}}^{(1)} + (-1)^l D_{\mathrm{ssB}}^{(2)} + D_{\mathrm{ssB}}^{(3)}\right), \tag{5.8}$$

where

$$D_{\mathrm{ssB}}^{(1)} = \int_0^\infty k^\mu f(k)\hat{A}_l(pk)\cos(2pk)\mathrm{d}k,$$
$$D_{\mathrm{ssB}}^{(2)} = \int_0^\infty k^\mu f(k)\hat{B}_l(pk)\sin(2pk)\mathrm{d}k, \quad D_{\mathrm{ssB}}^{(3)} = \int_0^\infty k^\mu f(k)\hat{C}_l(pk)\mathrm{d}k. \tag{5.9}$$

We interchange integration and summation, to find the finite Hankel series

$$D_{\rm ssB}^{(1)}(l,p,\mu;f)=\sum_{k=0}^{l}\frac{a_{2k}(l)}{p^{2k}}D_{\cos}(p,\mu-2k;f),$$
$$D_{\rm ssB}^{(2)}(l,p,\mu;f)=\sum_{k=0}^{l-1}\frac{b_{2k+1}(l)}{p^{2k+1}}D_{\sin}(p,\mu-2k-1;f),$$
$$D_{\rm ssB}^{(3)}(l,p,\mu;f)=\sum_{k=0}^{l}\frac{c_{2k}(l)}{p^{2k}}D_{0}(\mu-2k;f), \tag{5.10}$$

and $D_{\rm ssB}^{(2)}(l=0)=0$. We finally substitute series (5.10) into (5.8), to obtain the Bessel integral (5.1) as a finite linear combination of the functionals $D_{\cos,\sin,0}$ in (5.3)–(5.5). Since the argument $\mu-2k$ in (5.10) is usually negative, these functionals are understood as analytic continuations of the respective integrals, cf. Sect. 5.2. This is also the practical limitation of this method; it works for kernel functions $f$ where integral $D_{\exp}(p,\mu;f)$ in (5.2) can be calculated in closed form, in terms of special functions admitting a tractable analytic continuation in $\mu$. In Sect. 5.2, we will consider the modulated Gaussian $f(k)=\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$. Another possible kernel function is $f(k)=(1+ck)^{-\nu}\mathrm{e}^{-(b+\mathrm{i}\omega)k}$, so that the Bessel product in (5.1) is averaged with a double power law with exponential cutoff; in this case, $D_{\exp}(p,\mu;f)$ in (5.2) is an integral representation of a Kummer function. Also possible, a truncated modulated exponential $f(k)=\theta(k-\Lambda)\mathrm{e}^{-(b+\mathrm{i}\omega)k}$, where $\theta$ is the Heaviside step function and $D_{\exp}(p,\mu;f)$ an integral representation of the incomplete gamma function.

In Appendix B, we study integrals of type

$$D_{\rm ssB}^{(m,n)}(l,p,\mu;f)=\int_0^\infty k^{\mu+2}f(k)j_l^{(m)}(pk)j_l^{(n)}(pk)\mathrm{d}k, \tag{5.11}$$

where the superscripts $(m)$ and $(n)$ of the spherical Bessel functions denote multiple derivatives, and $D_{\rm ssB}^{(0,0)}=D_{\rm ssB}$. These integrals can be reduced to linear combinations of the integrals $D_{\rm ssB}(l,p,\mu;f)$ in (5.1).

### 5.2. Gaussian and Kummer averages

We calculate the Bessel integral

$$D_{\rm ssB}(l,p,\mu;a,b,\omega)=\int_0^\infty k^{\mu+2}\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}j_l^2(pk)\mathrm{d}k, \tag{5.12}$$

with integer index $l\geq 0$, positive scale parameter $p$, and real exponents $a>0, b$ and $\omega$. This integral converges for $\mu+2+2l>-1$, as the ascending series of $j_l^2(x)$ starts with $x^{2l}$, and $j_l(x)=\mathrm{O}(1/x)$, cf. (3.19) and (4.1). We proceed as outlined in Sect. 5.1, specifying $f(k)=\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$ in integrals (5.1)–(5.5).

The analytic continuation of $D_{\exp}(p,\mu;f)$ in (5.1) is effected by a parabolic cylinder function [4],

$$D_{\exp}(p,\mu;a,b,\omega)=\int_0^\infty k^{\mu}\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}\exp(2\mathrm{i}pk)dk$$
$$=\frac{\Gamma(\mu+1)}{(2a)^{(\mu+1)/2}}\mathrm{e}^{z^2/4}D_{-(\mu+1)}(z), \tag{5.13}$$

where $D_{-(\mu+1)}$ denotes the cylinder function, and we have introduced the variable

$$z = \frac{b + (\omega - 2p)\mathrm{i}}{\sqrt{2a}}. \tag{5.14}$$

In the case of Gaussian averages, $b$ is typically negative. We will also study the limit case $a = 0$ with positive $b$ in (5.12), that is, averages with Kummer distributions (power laws with modulated exponential cutoff $k^\mu \mathrm{e}^{-(b+\mathrm{i}\omega)k}$), cf. (5.19). The following also holds for complex $a$ with $\mathrm{Re}\, a > 0$, or imaginary $a$ and $b > 0$; in this case, we use the principal value of the root in (5.14). The Weber function $D_{-(\mu+1)}(z)$ is related to the confluent hypergeometric function by

$$D_{-(\mu+1)}(z) = \frac{\sqrt{\pi}}{2^{(\mu+1)/2}} \mathrm{e}^{z^2/4} \times \left[ \frac{1}{\Gamma(\mu/2+1)} {}_1F_1\left( \frac{-\mu}{2}, \frac{1}{2}, -\frac{z^2}{2} \right) - \frac{\sqrt{2}z}{\Gamma((\mu+1)/2)} {}_1F_1\left( \frac{1-\mu}{2}, \frac{3}{2}, -\frac{z^2}{2} \right) \right]. \tag{5.15}$$

By making use of the transformation $\mathrm{e}^{-x}{}_1F_1(a,b,x) = {}_1F_1(b-a,b,-x)$, we may write this as

$$D_{-(\mu+1)}(z) = \frac{\sqrt{\pi}\mathrm{e}^{-z^2/4}}{2^{(\mu+1)/2}} \left[ \frac{1}{\Gamma(\mu/2+1)} {}_1F_1\left( \frac{1+\mu}{2}, \frac{1}{2}, \frac{z^2}{2} \right) - \frac{\sqrt{2}\, z}{\Gamma((\mu+1)/2)} {}_1F_1\left( \frac{2+\mu}{2}, \frac{3}{2}, \frac{z^2}{2} \right) \right]. \tag{5.16}$$

We also note that $D_{-(\mu+1)}(z)$ is related to Kummer's confluent function by [4]

$$D_{-(\mu+1)}(z) = \frac{\mathrm{e}^{-z^2/4}}{2^{(\mu+1)/2}} U\left( \frac{\mu+1}{2}, \frac{1}{2}, \frac{z^2}{2} \right). \tag{5.17}$$

The Kummer function $U$ has a branch cut along the negative real axis, which makes this representation (5.17) less suitable for imaginary $z$ (i.e., $b = 0$ in (5.14)). The cylinder functions $D_{-(\mu+1)}$ degenerate into more elementary functions for half-integer index $\mu$ [4], but we do not assume this here.

Integral (5.13) is thus found as

$$D_{\exp}(p, \mu; a, b, \omega) = \frac{1}{2a^{(\mu+1)/2}} \times \left[ \Gamma\left( \frac{\mu+1}{2} \right) {}_1F_1\left( \frac{1+\mu}{2}, \frac{1}{2}, \frac{z^2}{2} \right) - \sqrt{2}z\Gamma\left( \frac{\mu+2}{2} \right) {}_1F_1\left( \frac{2+\mu}{2}, \frac{3}{2}, \frac{z^2}{2} \right) \right], \tag{5.18}$$

where $z = (b + \mathrm{i}(\omega - 2p))/\sqrt{2a}$. We also note the limit case $a = 0, b > 0$ in (5.12), where integral $D_{\exp}$ in (5.13) is elementary,

$$D_{\exp}(p, \mu; a = 0, b, \omega) = \frac{\Gamma(\mu+1)}{(b + \mathrm{i}(\omega - 2p))^{\mu+1}}. \tag{5.19}$$

This can also be obtained by using the asymptotic limit $U(\alpha, \beta, x) \sim x^{-\alpha}$ in (5.17).

The trigonometric integrals $D_{\cos,\sin,0}$ in (5.3)–(5.5) are assembled with $D_{\exp}$ in (5.18) or (5.19),

$$\begin{aligned} D_{\cos}(p, \mu; a, b, \omega) &= \frac{1}{2}(D_{\exp}(p, \mu; a, b, \omega) + D_{\exp}(-p, \mu; a, b, \omega)), \\ D_{\sin}(p, \mu; a, b, \omega) &= \frac{1}{2\mathrm{i}}(D_{\exp}(p, \mu; a, b, \omega) - D_{\exp}(-p, \mu; a, b, \omega)), \\ D_0(\mu; a, b, \omega) &= D_{\exp}(p = 0, \mu; a, b, \omega). \end{aligned} \tag{5.20}$$

$D_{\exp}$ and $D_{\cos,\sin,0}$ are analytic continuations in $\mu$ of the integrals in (5.3)–(5.5), which converge at the lower boundary only if $\mathrm{Re}\, \mu > -1$ or $-2$. This continuation is effected by the Weber function $D_{-(\mu+1)}$ in (5.13) or by the confluent functions in (5.18) or by (5.19). In case of integer exponents $\mu$, the gamma functions in (5.13), (5.18), and (5.19) can become singular, due to a negative integer argument $\mu - 2k$ in the series coefficients in (5.10), which requires $\varepsilon$ expansion to extract the singularities [7]. Numerically, it is more efficient to avoid $\varepsilon$ expansion and to use exponents close to integer and high-precision arithmetics; the latter is in any case necessary for high Bessel index $l$, as pointed out below. This method was used to calculate the Hankel entries in Tables 1, 2, and 3, since the $\varepsilon$ expansion of (5.18) involves a derivative of ${}_1F_1$ with regard to its first parameter, which slows the calculation down. In any case, $\varepsilon$ expansion can serve as a consistency check and is quite efficient if based on (5.19).

The Bessel integral (5.12) is thus calculated by substituting the trigonometric integrals (5.20) (with $D_{\rm exp}$ in (5.18)) into the three Hankel series (5.10), which become finite linear combinations of confluent functions, cf. (5.8),

$$D_{\rm ssB}(l,p,\mu;a,b,\omega) = \frac{1}{2p^2}\left((-1)^{l+1}D^{(1)}_{\rm ssB} + (-1)^l D^{(2)}_{\rm ssB} + D^{(3)}_{\rm ssB}\right). \tag{5.21}$$

The averaged Bessel derivatives $D^{(m,n)}_{\rm ssB}(l,p,\mu;f)$ in (5.11) with kernel function $f(k) = {\rm e}^{-ak^2-(b+{\rm i}\omega)k}$ can be reduced to linear combinations of $D_{\rm ssB}(l,p,\mu;a,b,\omega)$ in (5.21), cf. Appendix B.

For low and moderate Bessel index $l$, this method to evaluate the Bessel integral (5.12) is quite efficient and suitable for high-precision calculations, but becomes numerically cumbersome for high index, as the series coefficients $a_{2k}(l), b_{2k+1}(l)$ and $c_{2k}(l)$ in (5.10) become very large, cf. the end of Sect. 3, after (3.14). These coefficients are rational and do not depend on the parameters $(p,\mu,a,b,\omega)$ of the integrand in (5.12). At high $l$, the series $D^{(1)}_{\rm ssB}$ and $D^{(2)}_{\rm ssB}$ in (5.21) result in very large numbers (and to a lesser extend also series $D^{(3)}_{\rm ssB}$), which nearly cancel each other, so that high-precision arithmetics is required. ($2l$-digit precision at $l \sim 100$ is safe, for higher index it can be less than $2l$, but has to exceed $l$.) The Hankel series (5.10) are finite series without a proper expansion parameter to attenuate the large series coefficients. We may assume $p = 1$ in these series, by scaling $p$ into the parameters of the kernel function $f(k) = {\rm e}^{-ak^2-(b+{\rm i}\omega)k}$ in (5.12),

$$D_{\rm ssB}(l,p,\mu;a,b,\omega) = p^{-\mu-3}D_{\rm ssB}(l,1,\mu;a/p^2,b/p,\omega/p). \tag{5.22}$$

Numerically more efficient finite series expansions or closed expressions are only possible in special cases, such as $f(k) = {\rm e}^{-(b+{\rm i}\omega)k}, b \geq 0$, if paired with integer power-law exponent $\mu$ in (5.12) (Beltrami integrals [8]), or $f(k) = {\rm e}^{-ak^2}, {\rm Re}\, a \geq 0$, with even integer $\mu$ (Weber integrals [8]), or $f(k) = 1$ (Schafheitlin integrals [4]).

Instead of the Hankel representation (5.6), we may use the ascending series representation (3.19) of $j_l^2(pk)$ in integral (5.1). Term-by-term integration, effected by means of the integrals $D_0(\mu;a,b,\omega)$ in (5.20), then results in an asymptotic series [7], which is efficient if the exponents $a$ and/or $|b+{\rm i}\omega|$ are large and the Bessel index $l$ is kept moderate (Laplace/Fourier asymptotics). The integrals (5.12) occur in the multipole expansion of spherical Gaussian random fields [1], where the exponents of the kernel function $f(k)$ are moderate fitting parameters and the multipole index $l$ can become large, so that Laplace or Fourier asymptotics is not an option in this context. The high-$l$ asymptotics of the Bessel integrals (5.1), (5.11), and (5.12) is derived in Sects. 6 and 7, based on Nicholson's approximation of spherical Bessel functions.

## 6. Asymptotic approximations at high Bessel index

The goal is to find a numerically tractable approximation to the integral, cf. (5.11) and (5.12),

$$D^{(m,n)}_{\rm ssB}(l,p,\mu;a,b,\omega) = \int_0^\infty k^{\mu+2}{\rm e}^{-ak^2-(b+{\rm i}\omega)k} j_l^{(m)}(pk) j_l^{(n)}(pk){\rm d}k, \tag{6.1}$$

which can be used at high $l$, where the finite Hankel series derived in Sect. 5 become inefficient for the reasons mentioned after (5.21). More generally, we consider an arbitrary kernel function $g(k)$ and write

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g) = \int_0^\infty g(k) j_l^{(m)}(kp) j_l^{(n)}(kp) k^2 \mathrm{d}k. \tag{6.2}$$

The kernel $g(k)$ used here to define $D_{\mathrm{ssB}}^{(m,n)}$ differs from $f(k)$ in Sect. 5.1 by a factor $k^\mu$, $g(k) = k^\mu f(k)$, cf. (5.11) and Appendix B. The special case (6.1), the modulated Gaussian power law $g(k) = k^\mu \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$, will be discussed in Sects. 7 and 8.

We rescale the integration variable in (6.2) with $\alpha = (l+1/2)/p$,

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g) = \alpha^3 \int_0^\infty g(\alpha x) j_l^{(m)}((l+1/2)x) j_l^{(n)}((l+1/2)x) x^2 \mathrm{d}x, \tag{6.3}$$

and substitute the asymptotic high-$l$ limit of the Bessel product $j_l^{(m)} j_l^{(n)}$ derived in Appendix C. In the interval $0 \le x \le 1$, we can approximate $j_l^{(m)} j_l^{(n)} \approx 0$ and thus replace the lower integration boundary in (6.3) by 1. We then introduce a new integration variable $x = \sqrt{1+y}$ to find

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g) \sim \frac{\alpha^3}{2} \int_0^\infty g(\alpha\sqrt{1+y}) \times j_l^{(m)}((l+1/2)\sqrt{1+y}) j_l^{(n)}((l+1/2)\sqrt{1+y}) \sqrt{1+y}\, \mathrm{d}y. \tag{6.4}$$

The superscripts $(m)$ and $(n)$ denote multiple derivatives. To further proceed, we have to specify whether the index sum $m+n$ is even or odd, cf. Appendix C.3 and Sects. 6.2 and 6.3. In Sect. 6.1, we consider two special cases of even index sum, integrals containing the squares $j_l^2$ and $j_l'^2$.

### 6.1. High-$l$ asymptotics of integrals of type $\int_0^\infty g(k) j_l^2(kp) k^2 \mathrm{d}k$ and $\int_0^\infty g(k) j_l'^2(kp) k^2 \mathrm{d}k$

We start with integral $D_{\mathrm{ssB}}^{(0,0)}(l,p;g)$ in (6.2) and substitute the high-$l$ limit (C.9) of $j_l^2((l+1/2)\sqrt{1+y})$ into (6.4) to obtain

$$D_{\mathrm{ssB}}^{(0,0)}(l,p;g) \sim D_{\mathrm{ssB},\infty}^{(0,0)} = \frac{\alpha}{4p^2} \int_0^\infty g(\alpha\sqrt{1+y}) \frac{\mathrm{d}y}{\sqrt{y}}, \tag{6.5}$$

where $\alpha = (l+1/2)/p$. Integrals defining the asymptotics of $D_{\mathrm{ssB}}^{(m,n)}(l,p;g)$ in (6.2) are denoted by a subscript $\infty$; this distinction between $D_{\mathrm{ssB}}^{(m,n)}$ and its asymptotic high-$l$ limit $D_{\mathrm{ssB},\infty}^{(m,n)}$ will be useful in Sect. 6.3.

The high-$l$ approximation of integral $D_{\mathrm{ssB}}^{(1,1)}(l,p;g)$ in (6.2) is found by substituting the asymptotic limit of $j_l'^2((l+1/2)\sqrt{1+y})$ derived in (C.13) into (6.4),

$$D_{\mathrm{ssB}}^{(1,1)}(l,p;g) \sim D_{\mathrm{ssB},\infty}^{(1,1)} = \frac{\alpha}{4p^2} \int_0^\infty \frac{g(\alpha\sqrt{1+y})}{1+y} \sqrt{y}\, \mathrm{d}y. \tag{6.6}$$

In Sects. 6.2 and 6.3, we will use the same asymptotic technique, replacing the squares $j_l^2$ or $j_l'^2$ in the integrand by products of Bessel derivatives $j_l^{(m)} j_l^{(n)}$ with even and odd index sum $m+n$.

### 6.2. Integrals of type $\int_0^\infty g(k) j_l^{(m)}(kp) j_l^{(n)}(kp) k^2 \mathrm{d}k$ containing multiple derivatives of spherical Bessel functions with even index sum $m+n$

We calculate the high-$l$ asymptotics of integral $D_{\mathrm{ssB}}^{(m,n)}(l,p;g)$ in (6.2) for even index sum $m+n$, where $m$ and $n$ label multiple derivatives of the spherical Bessel functions. We start with approximation (6.4) of $D_{\mathrm{ssB}}^{(m,n)}(l,p;g)$ and substitute the asymptotic product $j_l^{(m)} j_l^{(n)}$ derived in (C.17) (valid for even index sum $m+n$) to obtain the high-$l$ limit

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g) \sim D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g), \tag{6.7}$$

$$D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g) = \frac{\alpha}{4p^2}(-1)^{(m+n)/2+mn} \int_0^\infty \frac{g(\alpha\sqrt{1+y})}{(1+y)^{(m+n)/2}} y^{(m+n-1)/2} \mathrm{d}y. \tag{6.8}$$

Here, we use the shortcut $\alpha = (l+1/2)/p$, and $m$ and $n$ are either both even or both odd non-negative integers. This covers the cases $m=n=0$ and $m=n=1$ in (6.5) and (6.6). The Bessel product $j_l^{(m)} j_l^{(n)}$ in integral (6.2) need not be a square, i.e. $m=n$ is not required, but $m+n$ has to be even for the asymptotic formula (6.7) to apply. The case of odd index sum, if one of the multiple derivatives in the product $j_l^{(m)} j_l^{(n)}$ is even and the other odd, will be studied in Sect. 6.3. The Bessel index only enters via $\alpha$ in (6.8), so that approximation (6.7) (as well as (6.21) below) remains valid for non-integer Bessel index $l$, cf. (2.1).

We will use the integrals $D_{\mathrm{ssB},\infty}^{(m,n)}$ defined in (6.8) exclusively for even index sum $m+n$; they satisfy the identity

$$D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g) = (-1)^j D_{\mathrm{ssB},\infty}^{(m+j,n-j)}(l,p;g). \tag{6.9}$$

If we put $j=(n-m)/2$, we find

$$D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g) = (-1)^{(n-m)/2} D_{\mathrm{ssB},\infty}^{((n+m)/2,(n+m)/2)}(l,p;g). \tag{6.10}$$

For instance, $D_{\mathrm{ssB},\infty}^{(0,2)} = -D_{\mathrm{ssB},\infty}^{(1,1)}$. In Sect. 6.3, we will also need the identity

$$\frac{\mathrm{d}}{\mathrm{d}p} D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g(k)/k) = -\frac{1}{p}[2D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g(k)/k) + D_{\mathrm{ssB},\infty}^{(m,n)}(l,p;g'(k))], \tag{6.11}$$

obtained by straight differentiation of (6.8), after replacing $g(k)$ by $g(k)/k$ in the integrand.

### 6.3. High-$l$ asymptotics of integral $\int_0^\infty g(k) j_l^{(m)}(kp) j_l^{(n)}(kp) k^2 \mathrm{d}k$ with odd index sum $m+n$

In (6.12) and (6.13), we explain the method to derive the high-$l$ limit of integral $D_{\mathrm{ssB}}^{(m,n)}$ in (6.2) for odd index sum $m+n$; in (6.14)–(6.21), we do the actual calculation. This integral is symmetric in $m$ and $n$, so that it suffices to consider $n>m$. If $m+n$ is odd and $n \ge m+3$, we use iteratively, cf. (6.2),

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) = \frac{\mathrm{d}}{\mathrm{d}p} D_{\mathrm{ssB}}^{(m,n-1)}(l,p;g(k)/k) - D_{\mathrm{ssB}}^{(m+1,n-1)}(l,p;g(k)), \tag{6.12}$$

and replace $D_{\mathrm{ssB}}^{(m,n-1)}$ by its asymptotic limit $D_{\mathrm{ssB},\infty}^{(m,n-1)}(l,p;g(k)/k)$ in (6.8), valid for even index sum $m+n-1$. We iterate this, until we arrive at $D_{\mathrm{ssB}}^{(\tilde{m},\tilde{m}+1)}(l,p;g(k))$, in which case we use, cf. (6.2),

$$D_{\mathrm{ssB}}^{(m,m+1)}(l,p;g(k)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p} D_{\mathrm{ssB}}^{(m,m)}(l,p;g(k)/k), \tag{6.13}$$

where we approximate $D_{\mathrm{ssB}}^{(m,m)}$ by $D_{\mathrm{ssB},\infty}^{(m,m)}(l,p;g(k)/k)$ in (6.8).

Performing the iteration of (6.12) $j-1$ times and making use of identity (6.9), we find

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) \sim j\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{(m,n-1)}(l,p;g(k)/k) + (-1)^j D_{\mathrm{ssB}}^{(m+j,n-j)}(l,p;g(k)). \tag{6.14}$$

Here, we choose $j=(n-m-1)/2$ and use (6.13) to obtain

$$\begin{aligned} D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) \sim & \frac{n-m-1}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{(m,n-1)}(l,p;g(k)/k) \\ & +(-1)^{(n-m-1)/2}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{((n+m-1)/2,(n+m-1)/2)}(l,p;g(k)/k). \end{aligned} \tag{6.15}$$

Finally, we substitute the identity, cf. (6.10),

$$D_{\mathrm{ssB},\infty}^{(m,n-1)}(l,p;g(k)/k) = (-1)^{(n-m-1)/2}D_{\mathrm{ssB},\infty}^{((n+m-1)/2,(n+m-1)/2)}(l,p;g(k)/k) \tag{6.16}$$

to find the asymptotic high-$l$ limit of integral $D_{\mathrm{ssB}}^{(m,n)}(l,p;g)$ in (6.2) as, cf. (6.8),

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) \sim (-1)^{(n-m-1)/2}\frac{n-m}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{((n+m-1)/2,(n+m-1)/2)}(l,p;g(k)/k), \tag{6.17}$$

which applies if the index sum $n+m$ is odd. This asymptotic formula is obviously symmetric in $n$ and $m$. We may again apply (6.16) to find the equivalent but less symmetric expression

$$D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) \sim \frac{n-m}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{(m,n-1)}(l,p;g(k)/k). \tag{6.18}$$

On the right-hand side of (6.17) and (6.18), the index sum is even, and the $p$ derivative is calculated in (6.11).

We list the asymptotic limit of two special cases of integral (6.2) with odd index sum $n+m$, assembled via (6.5), (6.6), and (6.11):

$$\begin{aligned} D_{\mathrm{ssB}}^{(0,1)}(l,p;g(k)) &\sim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{(0,0)}(l,p;g(k)/k) \\ &\sim -\frac{1}{4p^3}\int_0^\infty \frac{g(\alpha\sqrt{1+y})}{\sqrt{1+y}}\frac{\mathrm{d}y}{\sqrt{y}} - \frac{\alpha}{8p^3}\int_0^\infty g'(\alpha\sqrt{1+y})\frac{\mathrm{d}y}{\sqrt{y}} \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} D_{\mathrm{ssB}}^{(1,2)}(l,p;g(k)) &\sim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}D_{\mathrm{ssB},\infty}^{(1,1)}(l,p;g(k)/k) \\ &\sim -\frac{1}{4p^3}\int_0^\infty \frac{g(\alpha\sqrt{1+y})}{(1+y)^{3/2}}\sqrt{y}\mathrm{d}y - \frac{\alpha}{8p^3}\int_0^\infty \frac{g'(\alpha\sqrt{1+y})}{1+y}\sqrt{y}\mathrm{d}y. \end{aligned} \tag{6.20}$$

More generally, we can make the asymptotic formulas (6.17) and (6.18) more explicit by means of (6.8) and (6.11),

$$\begin{aligned} D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k)) \sim & (-1)^{(n-m+1)/2}\frac{n-m}{4p^3} \\ & \times\left(\int_0^\infty \frac{g(\alpha\sqrt{1+y})}{(1+y)^{(m+n)/2}}y^{(m+n-2)/2}\mathrm{d}y + \frac{\alpha}{2}\int_0^\infty \frac{g'(\alpha\sqrt{1+y})}{(1+y)^{(m+n-1)/2}}y^{(m+n-2)/2}\mathrm{d}y\right). \end{aligned} \tag{6.21}$$

This is the high-$l$ asymptotics of integral (6.2) with odd index sum $n+m$. For even index sum, we can approximate $D_{\mathrm{ssB}}^{(m,n)}(l,p;g(k))$ by $D_{\mathrm{ssB},\infty}^{(m,n)}$, as done in (6.7) and (6.8). In Sect. 7, we will specify the kernel function as modulated Gaussian power law, $g(k) = k^{\mu} e^{-ak^2-(b+i\omega)k}$, and calculate the high-$l$ asymptotics of integral $D_{\rm ssB}^{(m,n)}(l, p, \mu; a, b, \omega)$ in (6.1) by way of the explicit formulas (6.8) and (6.21).

## 7. Special cases: Bessel products averaged with Gaussian power laws and Kummer distributions

We study integral (6.2) with a modulated Gaussian power law as kernel function, $g_{\mu}(k; a, b, \omega) = k^{\mu} e^{-ak^2-(b+i\omega)k}$, cf. (6.1). We will usually not indicate the dependence on the real exponents $a > 0, b$ and $\omega$, writing $g_{\mu}(k)$ for this distribution; the exponent $\mu$ is assumed to be real as well, although complex $\mu$ can be considered. The limit case $a = 0$ with positive $b$ (Kummer distributions) is also admissible and so is complex $a$ with ${\rm Re}\, a > 0$ or ${\rm Re}\, a = 0$ with $b > 0$. This ensures convergence at the upper integration boundary of integral (6.1), since $j_l(x) = {\rm O}(1/x)$, cf. (4.1) and (4.6). The integral converges at the lower integration boundary for ${\rm Re}\, \mu > -3$, irrespectively of the non-negative integer indices $l, m$ and $n$, since $j_l(x) \propto x^l(1 + {\rm O}(x^2))$, cf. (3.18). For special index combinations, the condition $\mu > -3$ on the power-law exponent (required for $m = n = l = 0$) can be relaxed, but when performing multipole expansions of correlation functions on the unit sphere, the complete set of spherical Bessel functions is needed, including $j_0(x)$ [1].

Apparently, $g_{\mu}(k)/k = g_{\mu-1}(k)$, and

$$g'_{\mu}(k) = \mu g_{\mu-1}(k) - 2a g_{\mu+1}(k) - (b + i\omega) g_{\mu}(k), \tag{7.1}$$

which makes it possible to express the asymptotic limits $D_{\rm ssB,\infty}^{(m,n)}$ in (6.7), (6.11), (6.17), and (6.18) by the integrals, cf. (6.8),

$$I^{(m,n)}(\alpha; \mu, a, b, \omega) = \int_0^{\infty} \frac{g_{\mu}(\alpha\sqrt{1+y}; a, b, \omega)}{(1+y)^{(m+n)/2}} y^{(m+n-1)/2} {\rm d}y, \tag{7.2}$$

where $\alpha = (l + 1/2)/p$ is a positive parameter. In the following, we suppress the parameter dependence of $I^{(m,n)}$ on $a$,$b$ and $\omega$, writing $I^{(m,n)}(\alpha; \mu)$. A variable change $1 + y = (1 + t)^2, y = t(2 + t)$, in (7.2) gives

$$I^{(m,n)}(\alpha; \mu) = 2^{(m+n+1)/2} \int_0^{\infty} g_{\mu}(\alpha(1+t)) t^{(m+n-1)/2} \frac{(1+t/2)^{(m+n-1)/2}}{(1+t)^{m+n-1}} {\rm d}t, \tag{7.3}$$

where

$$g_{\mu}(\alpha(1+t)) = g_{\mu}(\alpha)(1+t)^{\mu} e^{-c_2 t^2 - c_1 t}, \quad g_{\mu}(\alpha) = \alpha^{\mu} e^{-a\alpha^2-(b+i\omega)\alpha},$$
$$c_2 = a\alpha^2, \quad c_1 = 2a\alpha^2 + (b + i\omega)\alpha, \quad \alpha = \frac{l + 1/2}{p}. \tag{7.4}$$

The numerical tests described in Sect. 8 and Tables 1, 2, and 3 are done in this parametrization of $I^{(m,n)}$. The integrals defined by (7.3) and (7.4) are numerically tame; high-precision arithmetics is not useful here, as the accuracy of the high-$l$ asymptotics derived in Sect. 6 is already limited by the Airy approximation of the Bessel functions, cf. Appendix C.

We can relate $D_{\rm ssB,\infty}^{(m,n)}(l, p; g_{\mu}(k))$ defined in (6.8) to $I^{(m,n)}(\alpha; \mu)$ in (7.2),

$$D_{\rm ssB,\infty}^{(m,n)}(l, p; g_{\mu}(k)) = \frac{\alpha}{4p^2}(-1)^{(m+n)/2+mn} I^{(m,n)}(\alpha; \mu). \tag{7.5}$$

This gives, for even index sum $m + n$, the high-$l$ asymptotics of integral $D_{\rm ssB}^{(m,n)}(l, p, \mu; a, b, \omega)$ in (6.1), according to (6.7).

To find the high-$l$ limit of integral (6.1) for odd index sum, we note, cf. (6.8),

$$D^{(m,n)}_{\mathrm{ssB},\infty}(l,p;g_\mu(k)/k) = \frac{\alpha}{4p^2}(-1)^{(m+n)/2+mn} I^{(m,n)}(\alpha;\mu-1), \tag{7.6}$$

since $g_\mu(k)/k = g_{\mu-1}(k)$. By making use of (7.1) and (7.5), we find

$$D^{(m,n)}_{\mathrm{ssB},\infty}(l,p;g'_\mu(k)) = \frac{\alpha}{4p^2}(-1)^{(m+n)/2+mn} \times[\mu I^{(m,n)}(\alpha;\mu-1) - 2aI^{(m,n)}(\alpha;\mu+1) - (b+\mathrm{i}\omega)I^{(m,n)}(\alpha;\mu)]. \tag{7.7}$$

On substituting (7.6) and (7.7) into (6.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}p}D^{(m,n)}_{\mathrm{ssB},\infty}(l,p;g_\mu(k)/k) = -\frac{\alpha}{4p^3}(-1)^{(m+n)/2+mn} \times[(\mu+2)I^{(m,n)}(\alpha;\mu-1) - 2aI^{(m,n)}(\alpha;\mu+1) -(b+\mathrm{i}\omega)I^{(m,n)}(\alpha;\mu)]. \tag{7.8}$$

In the identities (7.5)–(7.8), an even index sum $m+n$ is assumed. For odd index sum, the high-$l$ asymptotics of integral (6.1) is assembled via (6.17) or (6.18) and the derivative (7.8):

$$D^{(m,n)}_{\mathrm{ssB}}(l,p;g_\mu(k)) \sim \frac{\alpha}{4p^3}(-1)^{(n-m+1)/2}\frac{n-m}{2} \times[(\mu+2)I^{(m,n-1)}(\alpha;\mu-1) - 2aI^{(m,n-1)}(\alpha;\mu+1) - (b+\mathrm{i}\omega)I^{(m,n-1)}(\alpha;\mu)], \tag{7.9}$$

which applies for odd $m+n$. For even index sum, the counterpart to (7.9) reads, cf. (6.7) and (7.5),

$$D^{(m,n)}_{\mathrm{ssB}}(l,p;g_\mu(k)) \sim \frac{\alpha}{4p^2}(-1)^{(m+n)/2+mn} I^{(m,n)}(\alpha;\mu). \tag{7.10}$$

The integrals $I^{(m,n)}(\alpha;\mu)$ are defined in (7.3) and (7.4), with $\alpha = (l+1/2)/p$, and can readily be calculated numerically if the exponents of kernel $g_\mu(k) = k^\mu \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$ in (7.3) are moderate, cf. Sect. 8. The high-$l$ approximation of integral

$$D^{(m,n)}_{\mathrm{ssB}}(l,p;g_\mu(k)) = \int_0^\infty k^{\mu+2}\mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k} j_l^{(m)}(pk) j_l^{(n)}(pk)\mathrm{d}k \tag{7.11}$$

is thus settled, for odd as well as even index sum $m+n$, by the asymptotic equivalences (7.9) and (7.10), respectively.

## 8. Airy approximation versus Hankel expansion: a numerical comparison

In Tables 1, 2, 3 and Figs. 1, 2, we test the efficiency of the Airy approximations (7.9) and (7.10), by comparing with high-precision calculations of integral (7.11). The latter are performed with the Hankel decomposition in Sect. 5 and the recursive relations in Appendix B, where we identify the kernel function as $f(k) = \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$ to recover $D^{(m,n)}_{\mathrm{ssB}}$ in (7.11). As demonstrated in the tables, the high-$l$ approximations (7.9) and (7.10) are quite efficient even for very low Bessel index, but of limited accuracy. If high precision is required, the finite Hankel expansions in Sect. 5.2 are preferable, applicable at low and moderate $l$.

In the tables, the integrals (7.11) are denoted by $D^{(m,n)}_{\mathrm{ssB}}(l)$ and evaluated for two parameter sets $(p=1,\mu,a,b,\omega)$, cf. (5.22), which have been used in a multipole fit of the CMB temperature power spectrum [1]. In Table 1, we consider a real integrand with $\mu = 0, a = 6.26\times10^{-5}, b = -0.02$, and $\omega = 0$, so that the Bessel product in (7.11) is averaged with a Gaussian density. In Tables 2 and 3, we study the integrals (7.11) with a complex integrand defined by $\mu = 1, a = 0, b = 4.6\times10^{-3}$, and $\omega = 2.15\times10^{-2}$.

In this case, the spherical Bessel functions in (7.11) are averaged with a Kummer distribution, $k^{\mu}\mathrm{e}^{-(b+\mathrm{i}\omega)k}$, a power law with modulated exponential cutoff.

In each of the three tables, we list the integrals $D_{\mathrm{ssB}}^{(0,0)}(l)$, $D_{\mathrm{ssB}}^{(1,1)}(l)$, and $D_{\mathrm{ssB}}^{(0,1)}(l)$, cf. (B.1)–(B.3) with kernel $f(k) = \mathrm{e}^{-ak^2-(b+\mathrm{i}\omega)k}$, for several Bessel indices $l$ ranging between $l = 0$ and $10^4$, which is the multipole range of the CMB radiation presently measurable. First, the integrals (7.11) are calculated in Hankel expansion, by making use of (5.21), with the finite series (5.10) substituted. The series coefficients are compiled via (5.20) and (5.18) or (5.19). As for the derivatives $D_{\mathrm{ssB}}^{(1,1)}(l)$ and $D_{\mathrm{ssB}}^{(0,1)}(l)$, they are reduced to linear combinations of integrals of type $D_{\mathrm{ssB}}^{(0,0)}$, cf. (B.9) and (B.10), before the Hankel expansion (5.21) is performed. The same integrals are then calculated in Airy approximation (7.9) or (7.10) (with $I^{(m,n)}(\alpha;\mu)$ in parametrization (7.3)). Absolute and relative errors of the Airy approximation (as compared with the exact Hankel evaluation) of the integrals in Table 1 are depicted in Figs. 1 and 2 as a function of the Bessel index $l$.

For low and moderate Bessel indices, the finite Hankel series (5.10) can be used to calculate the integrals (7.11) in any desired precision; the digits indicated in the tables are all significant. At high $l$ ($l \gg 100$, say), the Hankel expansion becomes increasingly intractable for the reasons mentioned after (5.21). By contrast, the Airy approximation in (7.9) and (7.10) is designed for intermediate and high $l$, but is quite good at low Bessel index ($l \leq 10$) as well, cf. Figs. 1 and 2, and in any case easy to calculate. Its accuracy is evidently limited, no more than 3-digit precision is reached, but it suffices for CMB multipole fits, even at low $l$ [1].

## Appendix A: Products of Hankel polynomials

When squaring spherical Bessel functions in Sect. 3, we need a multiplication formula for Hankel polynomials (2.12) of the same degree, derived here. On multiplying two arbitrary polynomials of degree $n$ in $1/x$, we find

$$\sum_{m=0}^{n} \frac{a_m(n)}{x^m} \cdot \sum_{i=0}^{n} \frac{b_i(n)}{x^i} = \sum_{k=0}^{2n} \frac{c_k(n)}{x^k}, \quad c_k(n) = \sum_{m=\max(0,k-n)}^{\min(k,n)} a_m(n) b_{k-m}(n), \tag{A.1}$$

where we can replace $a_m b_{k-m}$ by $b_m a_{k-m}$, or use symmetrization, replacing $a_m b_{k-m}$ by $(a_m b_{k-m} + a_{k-m} b_m)/2$. As for the coefficients $c_k(n)$, we mention that $0 \leq k \leq 2n$ and

$$c_{k\leq n}(n) = \sum_{m=0}^{k} a_m(n) b_{k-m}(n), \tag{A.2}$$

where we use the notation $c_{k\leq n}(n)$ for $c_k(n)$ with $k \leq n$, and analogously for $k \geq n$,

$$c_{k\geq n}(n) = \sum_{m=k-n}^{n} a_m(n) b_{k-m}(n) = \sum_{m=0}^{2n-k} a_{m-n+k}(n) b_{n-m}(n). \tag{A.3}$$

Symmetrization can always be invoked in these product formulas for $c_k(n)$. We will occasionally drop the argument $n$ (polynomial degree), writing, for instance, $a_m$ for $a_m(n)$.

We split the product polynomial (A.1) into two polynomials with even and odd powers,

$$\sum_{k=0}^{2n} \frac{c_k(n)}{x^k} = \sum_{j=0}^{n} \frac{c_{2j}(n)}{x^{2j}} + \sum_{j=0}^{n-1} \frac{c_{2j+1}(n)}{x^{2j+1}}. \tag{A.4}$$

The even coefficients $c_{2j}(n), 0 \leq j \leq n$, can be written, for $2j \leq n$, as

$$c_{2j\leq n}(n) = \frac{1}{2}\sum_{m=0}^{2j}(a_m b_{2j-m} + a_{2j-m}b_m) = a_j b_j + \sum_{m=0}^{j-1}(a_m b_{2j-m} + a_{2j-m}b_m), \tag{A.5}$$

where we employed (A.2) (symmetrized) and the fact that terms with $m = l$ and $m = 2j - l$ are identical. We adopt the customary convention that a sum is void (and can thus be dropped) if the lower summation boundary exceeds the upper one. Negative upper summation boundaries can be avoided by replacing $a_j b_j + \sum_{m=0}^{j-1}$ by $-a_j b_j + \sum_{m=0}^{j}$. If $2j \geq n$, we use (A.3) to find

$$\begin{aligned} c_{2j\geq n}(n) &= \frac{1}{2}\sum_{m=0}^{2n-2j}(a_{2j-n+m}b_{n-m} + a_{n-m}b_{2j-n+m}) \\ &= a_j b_j + \sum_{m=0}^{n-j-1}(a_{2j-n+m}b_{n-m} + a_{n-m}b_{2j-n+m}), \end{aligned} \tag{A.6}$$

where we may replace $a_j b_j + \sum_{m=0}^{n-j-1}$ by $-a_j b_j + \sum_{m=0}^{n-j}$, if convenient.

Finally, we turn to the odd coefficients $c_{2j+1}(n), 0 \leq j \leq n-1$, in (A.4). If $2j + 1 \leq n$,

$$c_{2j+1\leq n}(n) = \frac{1}{2}\sum_{m=0}^{2j+1}(a_m b_{2j+1-m} + a_{2j+1-m}b_m) = \sum_{m=0}^{j}(a_m b_{2j+1-m} + a_{2j+1-m}b_m), \tag{A.7}$$

which follows from (A.2) and the fact that terms with $m = l$ and $m = 2j+1-l$ are identical. If $2j+1 \geq n$, we employ the symmetrized version of (A.3) to obtain

$$\begin{aligned} c_{2j+1\geq n}(n) &= \frac{1}{2}\sum_{m=0}^{2n-2j-1}(a_{2j+1-n+m}b_{n-m} + a_{n-m}b_{2j+1-n+m}) \\ &= \sum_{m=0}^{n-j-1}(a_{2j+1-n+m}b_{n-m} + a_{n-m}b_{2j+1-n+m}). \end{aligned} \tag{A.8}$$

The above multiplication formulas apply to arbitrary polynomials of the same degree. As for the Hankel polynomials in (2.12), it suffices to consider two special cases. First, we assume that the polynomial coefficients in the product formula (A.1) are related by $b_i(n) = \alpha(n)a_i(n)$, where $\alpha(n)$ is an arbitrary proportionality factor not depending on the summation index $i$. In this case, the coefficients (A.5)–(A.8) of the product polynomial in (A.1) simplify to

$$c_{2j\leq n}(n) = a_j b_j + 2\sum_{m=0}^{j-1} a_m b_{2j-m}, \quad c_{2j\geq n}(n) = -a_j b_j + 2\sum_{m=0}^{n-j} a_{2j-n+m}b_{n-m}, \tag{A.9}$$

for $0 \leq j \leq n$, and

$$c_{2j+1\leq n}(n) = 2\sum_{m=0}^{j} a_m b_{2j+1-m}, \quad c_{2j+1\geq n}(n) = 2\sum_{m=0}^{n-j-1} a_{2j+1-n+m}b_{n-m}, \tag{A.10}$$

for $0 \leq j \leq n-1$. In the second example, we assume the alternating relation $b_i(n) = (-1)^i\alpha(n)a_i(n)$ for the polynomial coefficients in (A.1). In this case, formulas (A.9) for the even coefficients still apply, but the odd coefficients in (A.4) vanish, $c_{2j+1}(n) = 0$.

## Appendix B: Normal form of integrals containing products of Bessel derivatives

The integrals $D_{\mathrm{ssB}}^{(m,n)}(l,p,\mu;f)$ in (5.11) can be reduced to the normal form $D_{\mathrm{ssB}}(l,p,\mu;f)$ defined in (5.1). We start with the integrals

$$D_{\mathrm{ssB}}^{(0,0)}(l,p,\mu;f) = \int_0^\infty k^{\mu+2} f(k) j_l^2(pk)\mathrm{d}k, \tag{B.1}$$

$$D_{\mathrm{ssB}}^{(1,1)}(l,p,\mu;f) = \int_0^\infty k^{\mu+2} f(k) {j'}_l^2(pk)\mathrm{d}k, \tag{B.2}$$

$$D_{\mathrm{ssB}}^{(0,1)}(l,p,\mu;f) = \int_0^\infty k^{\mu+2} f(k) j_l(pk) j_l'(pk)\mathrm{d}k, \tag{B.3}$$

where $f(k)$ is an arbitrary complex kernel function, $p$ a positive scale parameter, and $\mu$ a real exponent. We will express the integrals $D_{\mathrm{ssB}}^{(1,1)}$ and $D_{\mathrm{ssB}}^{(0,1)}$ as linear combinations of the normal form $D_{\mathrm{ssB}}^{(0,0)} = D_{\mathrm{ssB}}(l,p,\mu;f)$, which does not contain Bessel derivatives. In this appendix, it is not necessary to specify the kernel function $f(k)$ or to discuss convergence properties, but we note $j_l(x) \propto x^l(1+\mathrm{O}(x^2))$ and $j_l(x) = \mathrm{O}(1/x)$. The spherical Bessel functions constituting the products are of the same integer order $l \geq 0$, cf. (2.1). We replace the squared derivative ${j'}_l^2(kp)$ in (B.2) and the product $j_l(kp)j_l'(kp)$ in (B.3) by squares of spherical Bessel functions, substituting

$$\begin{aligned} {j'}_n^2(x) = j_n^2(x)\left(\frac{n^2}{x^2} - \frac{n}{2n+3}\right) \\ + j_{n+1}^2(x)\left(1 - \frac{n(2n+3)}{x^2}\right) + j_{n+2}^2(x)\frac{n}{2n+3} \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} j_n(x)j_n'(x) = j_n^2(x)\left(\frac{n}{x} - \frac{x}{2(2n+3)}\right) \\ - j_{n+1}^2(x)\frac{2n+3}{2x} + j_{n+2}^2(x)\frac{x}{2(2n+3)}. \end{aligned} \tag{B.5}$$

These two identities can readily be derived from the differentiation formula

$$j_n'(x) = \frac{n}{x} j_n(x) - j_{n+1}(x) \tag{B.6}$$

and the product $j_n(x)j_{n+1}(x)$ expressed as a sum of three squares,

$$\begin{aligned} j_n(x)j_{n+1}(x) = \frac{2n+3}{2x} j_{n+1}^2(x) \\ + \frac{x}{2(2n+3)} j_n^2(x) - \frac{x}{2(2n+3)} j_{n+2}^2(x). \end{aligned} \tag{B.7}$$

Identity (B.7) is obtained by squaring the recursive relation

$$j_{n+2}(x) = \frac{2n+3}{x} j_{n+1}(x) - j_n(x). \tag{B.8}$$

On substituting (B.4), we find integral $D_{\mathrm{ssB}}^{(1,1)}$ in (B.2) as

$$\begin{aligned} D_{\mathrm{ssB}}^{(1,1)}(l,p,\mu;f) = {} & \frac{l^2}{p^2} D_{\mathrm{ssB}}(l,p,\mu-2;f) - \frac{l}{2l+3} D_{\mathrm{ssB}}(l,p,\mu;f) \\ & + D_{\mathrm{ssB}}(l+1,p,\mu;f) - \frac{l(2l+3)}{p^2} D_{\mathrm{ssB}}(l+1,p,\mu-2;f) \\ & + \frac{l}{2l+3} D_{\mathrm{ssB}}(l+2,p,\mu;f), \end{aligned} \tag{B.9}$$

where $D_{\mathrm{ssB}} = D_{\mathrm{ssB}}^{(0,0)}$ is the normal form (B.1). Similarly, integral $D_{\mathrm{ssB}}^{(0,1)}$ in (B.3) can be written as linear combination of normal forms by means of identity (B.5),

$$\begin{aligned} D_{\mathrm{ssB}}^{(0,1)}(l,p,\mu;f) = {} & \frac{l}{p} D_{\mathrm{ssB}}(l,p,\mu-1;f) - \frac{p}{2(2l+3)} D_{\mathrm{ssB}}(l,p,\mu+1;f) \\ & - \frac{2l+3}{2p} D_{\mathrm{ssB}}(l+1,p,\mu-1;f) + \frac{p}{2(2l+3)} D_{\mathrm{ssB}}(l+2,p,\mu+1;f). \end{aligned} \tag{B.10}$$

Finally we consider integrals with products of multiple Bessel derivatives in the integrand, cf. (5.11), generalizing (B.1)–(B.3) as

$$D_{\mathrm{ssB}}^{(m,n)}(l,p,\mu;f) = \int_0^\infty k^{\mu+2} f(k) j_l^{(m)}(pk) j_l^{(n)}(pk) \mathrm{d}k, \tag{B.11}$$

where the superscripts denote $m$- and $n$-fold derivatives, and $j_l^{(0)} = j_l$. In this case, we employ the spherical Bessel equation,

$$j_n''(x) = \left( \frac{n(n+1)}{x^2} - 1 \right) j_n(x) - \frac{2}{x} j_n'(x), \tag{B.12}$$

to iteratively reduce the higher derivatives in (B.11) to zeroth and first order. Subsequently, we apply the reduction formulas (B.9) and (B.10) to express $D_{\mathrm{ssB}}^{(m,n)}$ as linear combination of the normal forms $D_{\mathrm{ssB}}$ in (B.1). For instance, we may substitute (B.12) for the second Bessel derivative in $D_{\mathrm{ssB}}^{(m,2)}(l,p,\mu;f)$ to obtain

$$\begin{aligned} D_{\mathrm{ssB}}^{(m,2)}(l,p,\mu;f) = {} & \frac{l(l+1)}{p^2} D_{\mathrm{ssB}}^{(m,0)}(l,p,\mu-2;f) \\ & - D_{\mathrm{ssB}}^{(m,0)}(l,p,\mu;f) - \frac{2}{p} D_{\mathrm{ssB}}^{(m,1)}(l,p,\mu-1;f). \end{aligned} \tag{B.13}$$

The integrals $D_{\mathrm{ssB}}^{(m,n)}$ in (B.11) are evidently symmetric in $m$ and $n$. If we put $m = 0$ in (B.13) and substitute (B.10), we obtain the reduction formula

$$\begin{aligned} D_{\mathrm{ssB}}^{(0,2)}(l,p,\mu;f) = {} & \frac{l(l-1)}{p^2} D_{\mathrm{ssB}}(l,p,\mu-2;f) - \frac{2l+2}{2l+3} D_{\mathrm{ssB}}(l,p,\mu;f) \\ & + \frac{2l+3}{p^2} D_{\mathrm{ssB}}(l+1,p,\mu-2;f) - \frac{1}{2l+3} D_{\mathrm{ssB}}(l+2,p,\mu;f). \end{aligned} \tag{B.14}$$

If we put $m = 1$ in (B.13) and substitute (B.9) and (B.10), we find

$$D_{\mathrm{ssB}}^{(1,2)}(l,p,\mu;f) = \frac{1}{p^3} l^2(l-1) D_{\mathrm{ssB}}(l,p,\mu-3;f)$$
$$-\frac{1}{p}\frac{l(5l+3)}{2(2l+3)} D_{\mathrm{ssB}}(l,p,\mu-1;f) + \frac{p}{2(2l+3)} D_{\mathrm{ssB}}(l,p,\mu+1;f)$$
$$-\frac{l(l-3)(2l+3)}{2p^3} D_{\mathrm{ssB}}(l+1,p,\mu-3;f) + \frac{2l-1}{2p} D_{\mathrm{ssB}}(l+1,p,\mu-1;f)$$
$$+\frac{1}{p}\frac{l(l-3)}{2(2l+3)} D_{\mathrm{ssB}}(l+2,p,\mu-1;f) - \frac{p}{2(2l+3)} D_{\mathrm{ssB}}(l+2,p,\mu+1;f). \tag{B.15}$$

If we put $m = 2$ in (B.13) and substitute (B.14) and (B.15), we arrive at the reduction formula

$$D_{\mathrm{ssB}}^{(2,2)}(l,p,\mu;f) = \frac{l^2(l-1)^2}{p^4} D_{\mathrm{ssB}}(l,p,\mu-4;f)$$
$$-\frac{1}{p^2}\frac{4l(l^2-1)}{2l+3} D_{\mathrm{ssB}}(l,p,\mu-2;f) + \frac{2l+1}{2l+3} D_{\mathrm{ssB}}(l,p,\mu;f)$$
$$+\frac{2}{p^4} l(l-1)(2l+3) D_{\mathrm{ssB}}(l+1,p,\mu-4;f) - \frac{4l+2}{p^2} D_{\mathrm{ssB}}(l+1,p,\mu-2;f)$$
$$-\frac{1}{p^2}\frac{2l(l-1)}{2l+3} D_{\mathrm{ssB}}(l+2,p,\mu-2;f) + \frac{2}{2l+3} D_{\mathrm{ssB}}(l+2,p,\mu;f). \tag{B.16}$$

This reduction to normal form is quite efficient for low Bessel index $l$ and can be used with $D_{\mathrm{ssB}}(l,p,\mu;f)$ in high-precision arithmetics, cf. Sect. 5, to check the high-$l$ Airy asymptotics of the integrals $D_{\mathrm{ssB}}^{(m,n)}$ in (B.11) derived in Sect. 6, cf. Tables 1, 2, and 3.

It should be noted that these reduction formulas require to calculate the normal forms $D_{\mathrm{ssB}}(l,p,\mu;f)$ in fairly high precision, since the individual terms in the linear combination cancel one another to a certain extend, resulting in precision loss, even though the terms are usually moderate. The Airy approximation of $D_{\mathrm{ssB}}(l,p,\mu;f)$ in Sect. 6 is not designed to distinguish $j_l$ from $j_{l+1}$ at high $l$, which is apparently necessary in these reduction formulas. It is thus preferable to calculate the normal form $D_{\mathrm{ssB}}(l,p,\mu;f)$ by means of the finite Hankel series in Sect. 5 when applying these reduction formulas, cf. (5.10) and (5.21). Moreover, the high-$l$ Airy approximation can directly be applied to the derivatives $D_{\mathrm{ssB}}^{(m,n)}$ in (B.11), without reducing them to normal form, cf. Section 6 and Appendix C, but it is less accurate than the Hankel series evaluation of the normal forms, cf. Tables 1, 2, 3 and Figs. 1, 2.

## Appendix C: Airy approximation of products of Bessel derivatives

### C.1 Nicholson asymptotics of the squared spherical Bessel function $j_l^2((l+1/2)x)$

We start with the uniform Nicholson approximation ($l \to \infty$) of $j_l((l+1/2)x)$, valid for positive argument $x$ [6],

$$j_l((l+1/2)x) \sim \sqrt{\pi}\left(\frac{\xi(x)}{1-x^2}\right)^{1/4} \frac{\mathrm{Ai}((l+1/2)^{2/3}\xi(x))}{(l+1/2)^{5/6}x^{1/2}}, \tag{C.1}$$

where the variable $\xi(x)$ is defined by

$$\xi(x \geq 1) = -\left(\frac{3}{2}\right)^{2/3} (\sqrt{x^2-1} - \arctan\sqrt{x^2-1})^{2/3}, \tag{C.2}$$

$$\xi(x \leq 1) = \left(\frac{3}{2}\right)^{2/3} (\operatorname{arctanh}\sqrt{1-x^2} - \sqrt{1-x^2})^{2/3}. \tag{C.3}$$

We note the derivative

$$\xi'(x) = -\frac{1}{x}\left(\frac{\xi(x)}{1-x^2}\right)^{-1/2}, \tag{C.4}$$

so that $\xi(1) = 0$ and $\xi'(1) = -2^{1/3}$. The squared Bessel function can thus be approximated by

$$j_l^2((l+1/2)x) \sim \frac{\pi}{x\lambda^5} \frac{|\xi(x)|^{1/2}}{|x^2-1|^{1/2}} \mathrm{Ai}^2(\lambda^2\xi(x)), \tag{C.5}$$

where we use the shortcut $\lambda = (l+1/2)^{1/3}$.

The squared Airy function admits the integral representation [9–13]

$$\mathrm{Ai}^2(z) = \frac{1}{2\pi^{3/2}} \int_0^\infty \cos\left(\frac{1}{12}t^3 + zt + \frac{\pi}{4}\right) \frac{dt}{\sqrt{t}}, \tag{C.6}$$

valid for real argument $z$. For large $z$, we invoke the Riemann–Lebesgue lemma to simplify the integrand in (C.6),

$$\mathrm{Ai}^2(z) \sim \frac{1}{2\pi^{3/2}} \int_0^\infty \cos(zt + \frac{\pi}{4}) \frac{dt}{\sqrt{t}} = \frac{1}{2\pi} \frac{1}{(-z)^{1/2}} \theta(-z), \tag{C.7}$$

where $\theta(z)$ is the Heaviside step function. In this approximation, $j_l^2((l+1/2)x)$ vanishes in the interval $0 \leq x < 1$, where $\xi(x)$ is positive, cf. (C.5). For $x \geq 1$, we introduce the variable $y = x^2-1, x = \sqrt{1+y}$ and note

$$\xi(\sqrt{1+y}) = -\left(\frac{1}{2}\right)^{2/3} \chi^2(y), \quad \chi(y) = 3^{1/3}(\sqrt{y} - \arctan\sqrt{y})^{1/3}, \tag{C.8}$$

where $\chi(y) \sim \sqrt{y}(1 - y/5 + \cdots)$. On substituting (C.7) and (C.8) into (C.5), the factor $\chi(y)$ drops out, and we obtain the high-$l$ limit

$$\sqrt{1+y}\, j_l^2((l+1/2)\sqrt{1+y}) \sim \frac{1}{2(l+1/2)^2} \frac{1}{\sqrt{y}}, \tag{C.9}$$

covering the interval $1 \leq x < \infty, y = x^2 - 1$.

### C.2 High-$l$ asymptotics of the squared derivative $j_l'^2((l+1/2)x)$

To find the uniform high-$l$ approximation of the derivative $j_l'((l+1/2)x)$ in leading order, we only need to differentiate the Airy function in Nicholson's formula (C.1) by making use of (C.4),

$$j_l'((l+1/2)x) \sim -\sqrt{\pi}\left(\frac{\xi(x)}{1-x^2}\right)^{-1/4} \frac{\mathrm{Ai}'((l+1/2)^{2/3}\xi(x))}{x^{3/2}(l+1/2)^{7/6}}. \tag{C.10}$$

Applying Airy's equation, $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$, and the large-$z$ approximation (C.7) of the squared Airy function, we find

$$\mathrm{Ai}'^2(z) = -z\mathrm{Ai}^2(z) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathrm{Ai}^2(z) \sim \frac{1}{2\pi}(-z)^{1/2}\theta(-z). \tag{C.11}$$

This approximation of $\mathrm{Ai}'^2$ is used to simplify the asymptotic squared derivative, cf. (C.10),

$$j_l'^2((l+1/2)x) \sim \frac{\pi}{x^3\lambda^7}\frac{\left|x^2-1\right|^{1/2}}{\left|\xi(x)\right|^{1/2}}\mathrm{Ai}'^2(\lambda^2\xi(x)), \tag{C.12}$$

where $\lambda = (l+1/2)^{1/3}$ and $\xi(x)$ is defined in (C.2) and (C.3). Accordingly, we approximate $j_l'^2((l+1/2)x) \approx 0$ in the interval $0 \le x < 1$. For $x \ge 1$, we introduce the variable $y = x^2 - 1$, and use (C.8) to obtain

$$(1+y)^{3/2} j_l'^2((l+1/2)\sqrt{1+y}) \sim \frac{\sqrt{y}}{2(l+1/2)^2}, \tag{C.13}$$

valid for positive $y$ and high $l$. Approximations tantamount to (C.9) and (C.13) can also be derived using Debye expansions or Legendre asymptotics, cf. [14] and references therein.

### C.3 Airy approximation of products of multiple derivatives $j_l^{(m)} j_l^{(n)}((l+1/2)x)$

To find the uniform asymptotic approximation of the $k$-fold derivative $j_l^{(k)}((l+1/2)x)$ in leading order, we only need to repeatedly differentiate the Airy function in (C.1), using $\xi'(x)$ in (C.4) and ignoring the $x$ dependence of all other factors in each step,

$$j_l^{(k)}((l+1/2)x) \sim (-1)^k\sqrt{\pi}\left(\frac{\xi(x)}{1-x^2}\right)^{1/4-k/2}\frac{\mathrm{Ai}^{(k)}((l+1/2)^{2/3}\xi(x))}{x^{1/2+k}(l+1/2)^{5/6+k/3}}. \tag{C.14}$$

The notation is the same as in the preceding subsections. $\mathrm{Ai}^{(k)}$ denotes the $k$-fold derivative of the Airy function. The Bessel index is denoted by $l$, and multiple derivatives are labeled by the indices $k, m$, and $n$, all non-negative integers.

Iterating Airy's equation, cf. after (C.10), we approximate $\mathrm{Ai}^{(2n)}(z) \sim z^n\mathrm{Ai}(z)$ for $z \to \infty$. We thus find, for even $m+n$ and real $z$,

$$\mathrm{Ai}^{(m)}(z)\mathrm{Ai}^{(n)}(z) \sim (-1)^{(m+n)/2+mn}\frac{1}{2\pi}(-z)^{(m+n-1)/2}\theta(-z), \tag{C.15}$$

where we used the asymptotic formulas (C.7) and (C.11). Squaring Nicholson's formula (C.14), we obtain

$$j_l^{(m)}((l+1/2)x) j_l^{(n)}((l+1/2)x) \sim \frac{(-1)^{m+n}\pi}{\lambda^{5+m+n}}\frac{\left|1-x^2\right|^{(m+n-1)/2}}{\left|\xi(x)\right|^{(m+n-1)/2}x^{1+m+n}} \times \mathrm{Ai}^{(m)}(\lambda^2\xi(x))\mathrm{Ai}^{(n)}(\lambda^2\xi(x)), \tag{C.16}$$

with $\lambda = (l+1/2)^{1/3}$ and $\xi(x)$ in (C.2) and (C.3). Here, we substitute the asymptotic Airy product (C.15). In the interval $0 \le x \le 1$, we can thus approximate $j_l^{(m)} j_l^{(n)} \approx 0$, as $\xi(x)$ is positive there. For $x \ge 1$, we introduce the variable $y = x^2 - 1$ and put $\xi(x) = -2^{-2/3}\chi^2(y)$ as in (C.8) to find the high-$l$ limit

$$(1+y)^{(1+m+n)/2} j_l^{(m)}((l+1/2)\sqrt{1+y}) j_l^{(n)}((l+1/2)\sqrt{1+y}) \sim \frac{(-1)^{(m+n)/2+mn}}{2(l+1/2)^2} y^{(m+n-1)/2}, \tag{C.17}$$

valid for even index sum $m+n$ and independent of the factor $\chi(y)$. In Sect. 6, we derive the high-$l$ asymptotics of integrals of type $\int_0^\infty g(k) j_l^{(m)}(kp) j_l^{(n)}(kp) k^2 \mathrm{d}k$, with unspecified kernel function $g(k)$, by employing approximation (C.17) for even index sum. The high-$l$ limit of these integrals with odd index sum $m + n$ can be reduced to integrals of the same type with even index sum, cf. Sect. 6.3.

## References


1. Tomaschitz, R.: Multipole fine structure of the cosmic microwave background: reconstruction of the temperature power spectrum. Mon. Not. Royal Astron. Soc. **427**, 1363–1383 (2012)
2. Newton, R.G.: Scattering Theory of Waves and Particles. Springer, New York (1982)
3. Jackson, J.D.: Classical Electrodynamics, 3rd edn. Wiley, New York (1999)
4. Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and Theorems for the Special Functions of Mathematical Physics. Springer, New York (1966)
5. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1996)
6. Olver F.W.J., Lozier D.W., Boisvert R.F., Clark C.W. (eds.): NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge (2010), http://dlmf.nist.gov
7. Tomaschitz, R.: Bessel integrals in epsilon expansion: Squared spherical Bessel functions averaged with Gaussian power-law distributions. Appl. Math. Comput. **225**, 228–241 (2013)
8. Tomaschitz, R.: Weber and Beltrami integrals of squared spherical Bessel functions: Finite series evaluation and high-index asymptotics. Math. Methods Appl. Sci. (2013), doi:10.1002/mma.2882
9. Laurenzi, B.J.: Moment integrals of powers of Airy functions. Z. Angew. Math. Phys. **44**, 891–908 (1993)
10. Reid, W.H.: Integral representations for products of Airy functions. Z. Angew. Math. Phys. **46**, 159–170 (1995)
11. Reid, W.H.: Integral representations for products of Airy functions, part 2, cubic products. Z. Angew. Math. Phys. **48**, 646–655 (1997)
12. Reid, W.H.: Integral representations for products of Airy functions, part 3, quartic products. Z. Angew. Math. Phys. **48**, 656–664 (1997)
13. Vallée, O., Soares, M.: Airy Functions and Applications to Physics, 2nd edn. Imperial College Press, London (2010)
14. Weinberg S.: Fluctuations in the cosmic microwave background. Phys. Rev. D **64**, 123512 (2001)


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