FOUNDATIONS OF CLASSICAL AND QUANTUM CHAOS IN EXTENDED ROBERTSON-WALKER COSMOLOGIES

Roman Tomaschitz

Dipartimento di Matematica Pura ed Applicata dell' Università degli Studi di Padova,
Via Belzoni 7, I-35131 Padova, Italy
and
Theoretical Physics Group, Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400 005, India

Abstract: A topological explanation for the recently observed anisotropy of the background radiation is given in terms of a cosmology that evolves from a totally chaotic initial state, and whose spacelike slices are infinite and multiply connected.

I. Isotropy and homogeneity, the cosmological postulates revisited

If one starts to do cosmology in the context of curved spaces, one is sooner or later compelled to make some assumptions about the metric, in fact to specify it, and not only the metric, but the space itself. It is hard to imagine to come to quantitative predictions otherwise.

Due to the fact that there does not seem to be an obvious choice, neither for the metric nor for the topology of the 3-space, one tries the simplest spaces first, which are the Euclidean space and the three-dimensional generalizations of a sphere and a hyperboloid. The distinguished feature of these cosmologies is that their Gaussian curvature is constant, independent of the space point, and the same for all geodesic surfaces through the point. These spaces are completely homogeneously and isotropically curved. The formal RW-space-time line element is then obtained by inserting in the Minkowski line element ds the line element $d\sigma$ of the curved 3-space, $ds^2 = -dt^2 + a^2(t)d\sigma^2$, where a(t) is the expansion factor that sets the length scale at a given instant of time. With the metric defined by this line element we obtain via the Einstein tensor the energy-momentum tensor in 'comoving coordinates', cf.[1,2]. An observer moving with the 'cosmic fluid' (which actually means that he does not move relative to the fluid) has his surroundings, namely the fluid, totally isotropic and homogeneous, the pressure and the density depending only on the cosmic time. This perfect homogeneity and isotropy of the 3-space manifests itself in the existence of a six-parameter group of isometries [2], one may think in terms of the rotations and translations of the Euclidean group, the first express the isotropy, the second the homogeneity. In fact, the 3-sphere (with or without antipodal points identified), the Minkowski hyperboloid, and evidently the Euclidean space exhaust all Riemannian spaces that

Group Theoretical Methods in Physics, XIX Int. Colloq., pp. 498-501,
edited by M. A. del Olmo *et al.*, CIEMAT, Madrid, 1992.

admit a six-parameter group of global isometries. Thus the existence of such a continuous group implies the constant curvature of the space, but goes far beyond. It is in fact a global rigidity requirement on the manifold, claiming that every observer moving with the fluid has the same surroundings, however large we may define it, e.g. by a (non-Euclidean) sphere of arbitrarily large radius.

In the next section we will consider RW-cosmologies which still are of constant curvature $-1/a^2(t)$, but have a multiply connected topology, and so do not admit a continuous group of isometries. Now, because we have a RW-line element, the energy-momentum tensor defined by the Einstein tensor is that of a perfectly homogeneous and isotropic fluid. Observers moving with it see the same surroundings within a sufficiently small hyperbolic sphere centred at the observer. The phrase 'sufficiently small' is necessary here, because a sphere may overlap with itself, depending on its size and location, if the space is multiply connected. Because in practice the danger to choose a sphere of observation that is overlapping with itself is rather small, such cosmologies appear in the perfect fluid/comoving observer approximation isotropic and homogeneous, like their simply connected counterpart, the Minkowski hyperboloid. I emphasize here 'perfect fluid approximation', since on the microscopic level, concerning classical and quantum geodesic motion of particles and rays, the space is definitively not homogeneous and isotropic, because of the appearance of a chaotic nucleus, cf.[3-6], the center of the manifold, a finite convex domain in the infinite 3-space. Particles trapped there exhibit the strongest form of chaos, the Bernoulli property. However, arbitrarily small perturbations, say of their momenta, will make them diffusing out of the nucleus into the manifold. Free particles move on regular but unstable trajectories. The same holds true for rays [4].

Quantum mechanically the situation depends very much on the actual size of the nucleus, which is determined by the Hausdorff dimension δ of the limit set of the covering group. If $\delta>1$, then massive particles admit a bound state wave field localized on the nucleus, which is stable, i.e. separated from the continuous spectrum.

On the other hand, the rays and trajectories on this nucleus constitute in a certain sense a set of measure zero, cf.[3,4]; therefore the classical Maxwell equations have the same spectrum as on the Minkowski hyperboloid, and the background radiation of quantized radiation fields is still perfectly isotropic in these multiply connected cosmologies [6].

II. Anisotropy and inhomogeneity: extended RW-cosmologies of constant negative curvature

The recently discovered large scale anisotropy in the temperature of the microwave background strongly suggests to relax the condition of homogeneity and isotropy. We have mentioned that a RW-line element always leads to a perfect Planck distribution of the frequencies, even if we impose a topology that breaks the six-dimensional continuous symmetry, and leads to inhomogeneities on a microscopic level. On the other hand, the multiply connected topology of the spacelike slices and the non-uniqueness of their metric, cf.[3-6], provides a natural way to extend RW-cosmologies, without giving up the condition of constant sectional Gaussian curvature.

We impose on the space-time manifold an extended RW- line element $ds^2 = -dt^2+a^2(t)d\sigma^2(x,t)$, where $d\sigma(x,t)$ is the line element of a metric $g_{ij}(x,t)$ on the 3-space that has constant curvature, say $-1/R^2$. The space of symmetric tensor fields g_{ij} on the topological 3-manifold that give rise to globally non-isometric spaces of sectional curvature $-1/R^2$ is finite dimensional, its dimension n depends on the topology of the manifold. One may think the fields g_{ij} parametrized by n independent parameters, and $g_{ij}(x,t)$ describes a time-dependent path in this deformation space.

The way to generate global anisotropies is not the only thinkable one. For example, one could add to the RW-metric on the Minkowski hyperboloid a small perturbation, $g_{ij}+\varepsilon h_{ij}$, but there is no indication how to choose the h_{ij}. Therefore it seems perfectly natural to keep the assumption of constant negative curvature alive, and to assume a multiply connected topology, that provides us with a finite dimensional deformation space of metrics. Clearly on the Minkowski hyperboloid we have n=1, and the metric is unique. For deformation spaces of fibering manifolds, their dimension, and explicit construction see [3,4].

The metric of the 3-space varies along a time-dependent path in this deformation space, which has a natural boundary. If the path reaches the boundary then the manifold changes the topology. For example, the genus of the fibers of the 3-space may change, a quasi-Fuchsian covering group degenerates to a Schottky group [6], or the manifold breaks to pieces, developing cusp singularities. The path continues in the deformation space of the new manifold till it reaches again its boundary.

The chaotic nucleus mentioned in Sec.1 has an obvious generalization. It is obtained like in the quasi-static case [3-6] from the convex hull $C_t(\Lambda)$ of the limit set $\Lambda(\Gamma)$ of the covering group Γ, the straight lines being those of the automorphic metric $g_{ij}(x,t)$ in the covering space H^3. Clearly the nucleus $C_t(\Lambda)\backslash\Gamma$ is now time dependent, and likewise the shapes of the covering geodesics in (H^3, g_{ij}), contrary to the quasi-static case with g_{ij} as the Poincaré metric.

There is then also some kind of universality, the qualitative features of the metric $g_{ij}(x,t)$ depending on whether t represents a point well inside the deformation space or close to its boundary. Moreover they depend on how fast $g_{ij}(x,t)$ varies. So, if the time-dependence of the g_{ij} is very weak, adiabatic so to say, the space-time metric is nearly isometric to a RW-metric.

But already in this case the temperature in the Planck distribution (still black-body) depends on the direction of the photon flux as well as on the space-coordinates [6].

A deformation sequence of the initial cosmic evolution could look like that. At $t = 0$ we start with a rigid hyperbolic 3-space manifold of finite volume, $\Lambda(\Gamma) = S_2$, $\delta(\Gamma) = 2$, which is densely filled with matter and radiation. For $t > 0$ it starts to decay, emitting radiation, simultaneously the vacuum, the infinite 3-space emerges as the vessel containing this radiation. The remnants of the primordial manifold appear as the chaotic nucleus of the 3-space, which itself passes through a sequence (n) of topologies of the type $I \times S_{g(n)}$, 3-spaces fibering over an interval, with Riemann surfaces as fibers. The genus g(n) of the fibers decreases during the evolution corresponding to a decrease of the Hausdorff dimensions δ_n of the limit sets of the covering groups Γ_n, $\delta_n \rightarrow 1$. During this period massive fields stay localized on the nucleus, until in a later deformation sequence δ_n drops below one, which happens for covering groups of the Schottky type [5,6]. Then the massive particles get finally delocalized, and move into the manifold created by the radiation field...

Another interesting question is that of particle creation-annihilation processes in quantum fields and backscattering processes in classical fields, during a period of rapid time-variation of the g_{ij} in the deformation space. Then the wave equations are not separable, and one cannot spot positive/negative frequencies during such periods, however one can strike the balance between two such periods of slow variation. In fact, even in the conformally coupled electromagnetic field, where the expansion factor a(t) always scales out, such processes can occur, in sharp contrast to the conventional RW-line elements [6].

Acknowledgment

The author acknowledges the support of the European Communities through their Science Programme under grant B/SC1*-915078 .

References

[1] C.W. Misner , K.S. Thorne, J.A. Wheeler, *Gravitation* (Freeman,San Francisco, 1973).
[2] S. Weinberg, *Gravitation and Cosmology* (Wiley, NY, 1972).
[3] R. Tomaschitz , J. Math. Phys. **32**, 2571(1991).
[4] R. Tomaschitz, Complex Systems **6**, 137 (1992).
[5] R. Tomaschitz, Intern. J. Theoret. Phys. **31**, 187 (1992).
[6] R.Tomaschitz, Dispersion and Topological Scattering in RW-Cosmologies I, II; TIFR-preprints, 1992, subm. to J. Math. Phys.