Electromagnetic radiation in multiply connected Robertson–Walker cosmologies

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Maxwell's equations on a topologically nontrivial cosmological background are studied. The cosmology is locally determined by a Robertson–Walker line element, but the spacelike slices are open hyperbolic manifolds, whose topology and geometry may vary in time. In this context the spectral resolution of Maxwell's equations in terms of horospherical elementary waves generated at infinity of hyperbolic space is given. The wave fronts are orthogonal to bundles of unstable geodesic rays, and the eikonal of geometric optics appears just as the phase of the horospherical waves. This fact is used to attach to the unstable geodesic rays a quantum mechanical momentum. In doing so the quantized energy-momentum tensor of the radiation field is constructed in a geometrically and dynamically transparent way, without appealing to the intricacies of the second quantization. In particular Planck's radiation formula, and the bearing of the multiply connected topology on the fluctuations in the temperature of the background radiation is discussed.

I. INTRODUCTION

This is the third part of a series[1,2] of papers on the global topological and geometrical structure of space–time, and its implications on classical and quantum mechanics. We are studying the global behavior of world lines, rays, particles, fields, classical or quantized, in an infinite space that has a nontrivial topological structure. And we figure out common features that do indeed exist between the classical and quantum description of objects moving in this space, which is itself a dynamical object, for its topology and geometry may vary in time.

Though contemporary cosmology originated from general relativity, the input that comes from this theory is rather small and qualitative: the existence of a space–time line element, the geodesic principle, the principle of general covariance. Additional assumptions are needed to fix the actual form of the line element: the principles of homogeneity and isotropy assure that the three-space is of constant curvature at any given instant of time. That determines the line element apart from a time-dependent expansion factor $a(\tau)$ that sets the length scale in the three-space, and apart of the sign k ($k=-1$, 0, or $+1$) of the curvature of the three-space. The line element can then be cast into the form $ds^2=-c^2\,d\tau^2+4a^2(\tau)(1+k|\mathbf{x}|^2/R^2)^{-2}\,dx^2$. That is just the Minkowski metric with Euclidean three-space replaced by the $a(\tau)$-scaled, constantly curved three-space. Such line elements are usually referred to as of Robertson–Walker (RW) type.

What have Einstein's equations then to contribute to cosmology? Clearly they do not determine the metric, even not locally, because for that we would have to know the energy-momentum tensor of the universe, which is not the case. If we insert the RW metric into the Einstein equations we can however express the energy and the pressure density of the universe as a function of $a(\tau)$ and its derivatives. The requirement of positivity of energy and pressure gives then restrictions on the asymptotic behavior of $a(\tau)$, for example, in the case $k=-1$ positivity would require $a(\tau)\sim\tau$ for $\tau\to\infty$. In the limit $\tau\to 0$ (or $\tau\to-\infty$, if one believes in an infinite past) positivity arguments of this kind are not reliable, because in this limit either

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pressure or energy density diverge to infinity, and the classical Einstein equations are not supposed to give even qualitatively correct results in this regime. Anyhow, the expansion factor, the pressure, and the energy density remain unknown functions, and even if one postulates *ad hoc* and in addition a universal thermodynamic equation of state this situation is not remedied. I do not want to deny here the considerable heuristic and phenomenological value of such supplements, however we are here more interested in the underlying structure of space–time, and on its bearing on the microscopic dynamics, and about that these equations have little to say.

Up to now we have only addressed the local, metric structure, the line element. In cosmology a much more important question is the global structure of space–time, the global topology, which is clearly left totally undecided by general relativity, being a purely metrical theory. Usually this question is settled in textbooks by postulating maximal symmetry, the existence of a continuous six-parameter symmetry group transitively acting on the spacelike slices. This postulate restricts then the possible topologies to be either Euclidean space ($k=0$), or the three-sphere ($k=1$), or projective three-space ($k=1$), or a shell of the Minkowski hyperboloid ($k=-1$). Moreover there is only one metric of constant curvature that one can impose on these topologies. Maximal symmetry implies perfect homogeneity and isotropy.

This principle of maximal symmetry of the three-space is of course very convenient, because it fixes once and for all both the topology and the geometry. However we regard it as too rigid, too static, unnecessary restrictive, and we suggest to relax it by requiring instead of maximal symmetry constant (negative) curvature. Moreover we assume that the three-space is infinite. Then the topology does not any more determine the metric uniquely, which therefore may itself vary in time, apart from the scaling with the expansion factor. Locally, on sufficiently small coordinate patches we have still the six-parameter group of isometries, and the space is locally homogeneous and isotropic. For a more pictorial description of isotropy, homogeneity, and symmetry in this context we refer to Ref. 3. Finally I mention that as soon as the topology is nontrivial there does not any more exist a continuous group of global isometries, in a sense it is replaced by the discrete covering group of the three-space manifold.

Because the actual impact of general relativity is rather small, one has to pose the question in how far Riemannian geometry enters at all in cosmologies with constantly curved spaces. Though it provides the underlying structure of the principle of general relativity, the actual use of Riemannian geometry is trivialized by the simplicity of the RW line element. At first, the time coordinate in the line element is extremely distinguished, which highly encourages the use of three-dimensional formalism, using time as a parameter. The second reason is that the constant sectional Gaussian curvature of the three-space makes the use of the Riemann curvature tensor superfluous, and renders the three-space more to an object of elementary geometry. This is perhaps the main reason that Newtonian theories of cosmology still enjoy a certain popularity.

The paper is organized as follows. In Secs. II and III the spectral theory of Maxwell's equations in infinite, multiply connected spaces is on the agenda. The spectrum and the eigenfunctions are explicitly calculated, and the orthogonality and completeness relations are derived. Many of the technicalities used in the spectral theory of the scalar wave equations in Refs. 1 and 2 have been employed here without further mentioning, likewise most of the notation. In particular this holds true for Sec. II, where the spectral theory in the Poincaré upper half-space H^3, isometric to the Minkowski hyperboloid, is developed. In Sec. IV nothing new happens, but I hope it enhances a little the understanding of electromagnetism in RW cosmologies. Maxwell's equations and conservation laws in terms of **E** and **B** fields on the spacelike slices are formulated.

In Sec. V we discuss the Planck radiation formula for multiply connected RW cosmologies, employing the close relation between wave mechanics and the unstable classical mechanics in RW cosmologies.[1,2] In the case of the electromagnetic field it happens that the eikonal of ray optics appears as the phase in the elementary waves that we identified in Secs. II and III as the eigenfunctions. This provides an exact and concrete geometric content of the Einstein formula $p_\mu=\hbar k_\mu$, which we use to attach to the rays a definitive quantum mechanical momentum. This enables us then to construct the quantized energy-momentum tensor in a painless and unambiguous way. Finally we close Sec. VI with some comments on the recently discovered fluctuations in the microwave background, and demonstrate how they fit into the cosmic scene advocated here. For further discussion we refer to the Conclusion, Sec. VI.

II. THE GENERAL SETTING: MAXWELL's EQUATIONS IN THE POINCARÉ HALF-SPACE H^3

To avoid the use of unduly heavy Riemannian geometry like three-indices and the Ricci tensor, it is convenient to adopt the following form of Maxwell's equations,

$$\frac{\partial}{\partial x^\kappa}\left[\sqrt{-g}g^{\lambda\nu}g^{\kappa\mu}\left(\frac{\partial A_\mu}{\partial x^\nu}-\frac{\partial A_\nu}{\partial x^\mu}\right)\right]=0, \tag{2.1}$$

$$\frac{\partial}{\partial x^\lambda}(\sqrt{-g}g^{\lambda\nu}A_\nu)=0, \quad A_0=0. \tag{2.2}$$

We will use only coordinate transformations that are time-independent, and the Coulomb gauge (2.2) is covariant with respect to them. Most of the notation used here concerning hyperbolic geometry has been defined in Refs. 1 and 2, which are the prerequisite for this paper, otherwise it is self-contained. Greek indices run from 0 to 3, Latin from 1 to 3. We write $x^\mu=(\tau,x)$, very often we will use less conventional complex notation, $x^\mu=(\tau,z,t)$, $z=y_1+iy_2$, always $g_{00}=-c^2$, $g_{ik}=h^2a^2(\tau)\delta_{ik}$, $g_{0i}=0$. For h, that specifies the three-space metric, we will use mostly $h=R/t$ in the Poincaré half-space model H^3, or $h=2/(1-|x|^2/R^2)$ in the B^3 model, cf. Refs. 4 and 5.

Because of the special form of the metric $g_{\mu\nu}$, and because of Eq. (2.2), the zero component of the system (2.1) is identically satisfied. In (2.1) and (2.2) we make a variable separation, $A_i=\varphi(\tau)\hat{A}_i(\mathbf{x})$, with the separation constant λ^2/R^2, and obtain

$$\frac{\lambda^2}{R^2}\hat{A}_i=\frac{1}{h}\frac{\partial}{\partial x^k}\left[\frac{1}{h}\left(\frac{\partial\hat{A}_i}{\partial x^k}-\frac{\partial\hat{A}_k}{\partial x^i}\right)\right], \tag{2.3}$$

$$\frac{\partial}{\partial x^k}(h\hat{A}_k)=0, \quad \varphi^\pm(s,\tau)=\exp\left(\mp is\Lambda\int_{\tau_0}^{\tau}a^{-1}(\tau)d\tau\right), \tag{2.4}$$

the $\varphi^\pm$ constitute the two fundamental solutions of the equation for the time component, and we have put $\lambda=-is$, $\Lambda:=c/R$.

Next we solve Eqs. (2.3) and (2.4) in the H^3 model with $h=R/t$. The generalized eigenfunctions of the scalar wave equation[1,2] are powers P^λ of the Poisson kernel

$$P(z,t,\xi)=\frac{tR}{(|z-\xi|^2+t^2)}. \tag{2.5}$$

The λ, $\xi\in\mathbb{C}$ are spectral variables.

The wave fronts, $P^\lambda(z,t,\xi)=\text{const}$, with λ, ξ fixed, are horospheres.[2] If $\xi=\infty$, we can take $P^\lambda(\xi=\infty)=(t/R)^\lambda$. In this case the wave fronts are just Euclidean planes in H^3, parallel to the boundary $\mathbb{C}$. For further details we refer to the Appendix of Ref. 2.

It is now easily verified that the covariant three-vectors $\mathbf{a}^+(t,\lambda,\xi=\infty):=(1,0,0)\ (t/R)^\lambda$, $\mathbf{a}^-(t,\lambda,\xi=\infty):=(0,-1,0)(t/R)^\lambda$ satisfy Eq. (2.3) and the tansversality condition (2.4), since they are tangent to the wave fronts. We complement the $\mathbf{a}^\pm$ by the vector $\mathbf{a}^S(t,\lambda,\xi=\infty)$ $:=(0,0,1)\ (t/R)^\lambda$ to obtain an orthogonal triad.

The Lorentz group acting as three-dimensional Möbius transformations[4,5] in H^3 is the invariance group of the g_{ij}, and thus also the invariance group of equations (2.3) and (2.4). Therefore, if we apply Möbius transformations to the vectors $\mathbf{a}^\pm$, they remain solutions of Eqs. (2.3) and (2.4), and it turns out that we get in this way sufficiently many generalized eigenfunctions to compose a complete set of transversal vector fields.

In fact, let be $\alpha_\xi(z)=(z-\xi)^{-1}$, $\xi\in\mathbb{C}$, a Möbius transformation acting in the complex plane. Its lift to H^3 is[4]

$$\alpha_\xi(z,t)=\left(\frac{\bar{z}-\bar{\xi}}{|z-\xi|^2+t^2},\frac{t}{|z-\xi|^2+t^2}\right). \tag{2.6}$$

We use complex notation, $(z,t)\leftrightarrow(y_1,y_2,t)\in H^3$, and we denote by $[\alpha'_\xi(z,t)]$ the Jacobi matrix. We apply Eq. (2.6) to the $\mathbf{a}^\pm$, $\mathbf{a}^S$, and obtain

$$\mathbf{a}^+(z,t,\lambda,\xi)=(1,0,0)\,[\alpha'_\xi(z,t)]P^\lambda(z,t,\xi), \tag{2.7}$$

and analogously for $\mathbf{a}^-$, $\mathbf{a}^S$.

It will be more convenient later, cf. Eq. (3.9), to use instead of the $\mathbf{a}^\pm$ another basis for the transversal vector fields, namely, $\mathbf{a}^R:=-2^{-1/2}a(\tau)(\mathbf{a}^+-i\mathbf{a}^-)$, and $\mathbf{a}^L:=-2^{-1/2}a(\tau)(\mathbf{a}^+ +i\mathbf{a}^-)$, $a(\tau)$ is the expansion factor in the RW line element. The Jacobian in Eq. (2.7) is readily calculated, and we have

$$\begin{aligned}\mathbf{a}^R(z,t,\lambda,\xi)&=-2^{-1/2}a(\tau)(1,i,0)\,[\alpha'_\xi(z,t)]P^\lambda(z,t,\xi)\\ &=2^{-1/2}a(\tau)\overline{\left((z-\xi)^2-t^2,i\,(z-\xi)^2+it^2,2t(z-\xi)\right)}\frac{P^\lambda(z,t,\xi)}{(|z-\xi|^2+t^2)^2},\\ \mathbf{a}^L&=\overline{\mathbf{a}^R(z,t,\bar{\lambda},\xi)}=2^{-1/2}a(\tau)(-1,i,0)\,[\alpha'_\xi(z,t)]P^\lambda(z,t,\xi),\\ \mathbf{a}^S(z,t,\lambda,\xi)&=a(\tau)(0,0,1)\,[\alpha'_\xi(z,t)]P^\lambda(z,t,\xi)\\ &=a(\tau)(-2\,\mathrm{Re}(z-\xi)t,-2\,\mathrm{Im}(z-\xi)t,|z-\xi|^2-t^2)\frac{P^\lambda(z,t,\xi)}{(|z-\xi|^2+t^2)^2}.\end{aligned} \tag{2.8}$$

We define a vectorial scalar product on the spacelike sections,

$$\langle\mathbf{a},\mathbf{b}\rangle:=g^{ij}a_i\bar{b}_j,\quad g^{ij}:=a^{-2}(\tau)\delta_{ij}t^2/R^2. \tag{2.9}$$

Then we obtain

$$\langle\mathbf{a}^X(z,t,\lambda,\xi),\mathbf{a}^Y(z,t,\lambda',\xi')\rangle=P^{1+\lambda}(z,t,\xi)P^{1+\overline{\lambda'}}(z,t,\xi')\delta_{XY}+O(|\xi-\xi'|), \tag{2.10}$$

where X,Y stand for R, L, S. We define a Hilbert space scalar product for these vector fields as

$$\langle\mathbf{a},\mathbf{b}\rangle_{H^3}:=\int_{H^3}d\mu_{H^3}(z,t)\langle\mathbf{a}(z,t),\mathbf{b}(z,t)\rangle, \tag{2.11}$$

with the volume element of the three-space

$$d\mu_{H^3}=a^3(\tau)R^3/t^3\,dy_1\,dy_2\,dt \quad (z=y_1+iy_2). \tag{2.12}$$

Comparison with the scalar case suggests to take $\lambda=-is$, $s\in\mathbb{R}$. For then we have, cf. the Appendix of Ref. 2,

$$\langle \mathbf{a}^X(s,\xi),\mathbf{a}^Y(s',\xi')\rangle_{H^3}=a^3(\tau)\delta_{XY}\int_{H^3}P^{1-is}(z,t,\xi)P^{1+is'}(z,t,\xi')d\mu_{H^3}$$
$$=2\pi^3a^3(\tau)R^5s^{-2}\delta(\xi-\xi')\delta(s-s')\delta_{XY}. \tag{2.13}$$

Finally, with the spectral measure

$$d\sigma_{H^3}(s,\xi)=\frac{s^2\,ds\,d^2\xi}{a^3(\tau)4\pi^3R^5} \tag{2.14}$$

as in the scalar case, cf. A(6) of Ref. 2, we obtain the completeness relation [we put $\lambda=-is$ in Eq. (2.8)],

$$\int_{\mathbf{R}^3}d\sigma_{H^3}(s,\xi)\,[a_i^R(z,t,s,\xi)\overline{a_j^R(z',t',s,\xi)}+R\to L+R\to S]=g_{ij}\delta_{\mu_{H^3}}(z,t;z',t'). \tag{2.15}$$

$\delta_{\mu_{H^3}}$ is the δ function with respect to the measure Eq. (2.12). In fact, Eq. (2.15) can be easily reduced to the scalar completeness relation, using

$$[a_i^R(z,t,s,\xi)\overline{a_j^R(z',t',s,\xi)}+R\to L+R\to S]$$
$$=a^2(\tau)\delta_{ij}\frac{R^2}{tt'}P^{1-is}(z,t,\xi)P^{1+is}(z',t',\xi)+O(|z-z'|+|t-t'|). \tag{2.16}$$

We have shown now that the $\mathbf{a}^{R,L,S}$ in Eq. (2.8) with $\lambda=-is$, $s\in\mathbb{R}$, $\xi\in\mathbb{C}$, constitute a complete orthogonal set in the space of square-integrable complex-valued three-vector fields on H^3. The operator on the right-hand side of Eq. (2.3) is Hermitian with respect to Eq. (2.11) in the subspace of the transversal fields, generated by the $\mathbf{a}^{R,L}$, its generalized eigenfunctions.

Credit should be given here to Ref. 6, where wave equations for vector and tensor fields in RW geometries have been studied. There the spectral resolution was carried out in terms of pseudospherical harmonics on the Minkowski hyperboloid. The formalism adopted in this section is however more suitable for multiply connected hyperbolic spaces, where we do not have a continuous symmetry group acting upon.

III. SPECTRAL THEORY OF MAXWELL's EQUATIONS ON A MULTIPLY CONNECTED HYPERBOLIC THREE-MANIFOLD *F*

A. Automorphic vector fields

To solve the spectral problem (2.3) and (2.4) on F we have to look for vector fields $\mathbf{a}^\Gamma(\mathbf{x})$ in the covering space H^3 that are invariant under coordinate transformations effected by elements (α) of the covering group Γ. The basic concepts of the covering space construction, like the fundamental polyhedron F, the group Γ generated by its face-identifying Möbius transformations, the limit set $\Lambda(\Gamma)$, its convex hull $C(\Lambda)$, and the quotient $C(\Lambda)\setminus\Gamma$ have been extensively discussed in Refs. 1, 2, 7, and 8.

Invariance under Γ means

$$a_j^\Gamma(x) = a_i^\Gamma(\alpha x)[\alpha' x]_j^i, \quad \text{for all} \quad \alpha \in \Gamma, \tag{3.1}$$

$[\alpha'(x)]$ denotes the Jacobian of α like in Eq. (2.7).

We generate invariant vector fields by periodization: If $a_i(x)$ is a vector field on H^3, then its covering projection onto the polyhedron F is

$$a_j^\Gamma(x) = \sum_{\gamma \in \Gamma} a_i(\gamma x)[\gamma' x]_j^i, \tag{3.2}$$

or contravariantly,

$$a^{\Gamma j}(x) = \sum_{\gamma \in \Gamma} [\gamma' x]_i^{-1j} a^i(\gamma x). \tag{3.3}$$

In the following we will often suppress the indices, thinking in terms of matrix multiplication and line and row vectors.

We also mention here, though it will not be relevant in the following, that if there is a subgroup Γ' of Γ, whose elements leave $a_i(x)$ invariant, cf. Eq. (3.1), then we have to take the summation in Eqs. (3.2) and (3.3) over a system of representatives of the right cosets of Γ', and not over the whole group Γ. Compare, for example, the Poincaré series for hyperbolic manifolds with parabolic cusp singularities in Ref. 9. In the case of the Poincaré metric we have $\Gamma' = \Gamma$, and no periodization at all is needed.

B. Vectorial Eisenstein series

Let $\mathbf{a}$ be a covariant vector field defined analogously to the $\mathbf{a}^{L,R,S}$ in Eq. (2.8) as

$$\mathbf{a}(z,t,\lambda,\xi) := a(\tau)(c_1,c_2,c_3)[\alpha_\xi'(z,t)]P^\lambda(z,t,\xi), \tag{3.4}$$

the c_i constitute a constant complex line vector. The periodization of Eq. (3.4) according to Eq. (3.2) is the Eisenstein series

$$\mathbf{a}^\Gamma(z,t,\lambda,\xi,c_i) := \sum_{\gamma \in \Gamma} a(\tau)(c_1,c_2,c_3)[\alpha_\xi'(\gamma(z,t))][\gamma'(z,t)]P^\lambda(\gamma(z,t),\xi). \tag{3.5}$$

This is a straightforward generalization of scalar Eisenstein series.[9–11] Let γ be a Möbius transformation $z \to (az+b)/(cz+d)$, $ad-bc=1$, and be $\alpha_\xi(z) = (z-\xi)^{-1}$, $\xi \in \mathbb{C}$. Then we have

$$\alpha_\xi \circ \gamma = L_\gamma \circ \alpha_{\gamma^{-1}\xi}, \tag{3.6}$$

with $L_\gamma(z) = \gamma^{-1\prime}(\xi)z + c/(-c\xi+a)$. The $\gamma^{-1\prime}$ denotes here of course the complex derivative of γ^{-1}. Clearly Eq. (3.6) lifts to H^3 as it stands, with $\alpha_\xi(z,t)$ as in Eq. (2.6) and

$$L_\gamma(z,t) = (L_\gamma(z), |\gamma^{-1\prime}\xi|\, t). \tag{3.7}$$

The Jacobian $[L_\gamma']$ of Eq. (3.7) is independent of (z,t), and using the chain rule for Jacobians and Eq. (A32) in Ref. 2, we may write Eq. (3.5) as

$$\mathbf{a}^\Gamma(z,t,\lambda,\xi) = a(\tau) \sum_{\gamma \in \Gamma} |\gamma^{-1\prime}\xi|^\lambda (c_1 \operatorname{Re}(\gamma^{-1\prime}\xi) + c_2 \operatorname{Im}(\gamma^{-1\prime}\xi),$$

$$-c_1 \operatorname{Im}(\gamma^{-1\prime}\xi) + c_2 \operatorname{Re}(\gamma^{-1\prime}\xi), c_3 |\gamma^{-1\prime}\xi|)[\alpha_{\gamma^{-1}\xi}'(z,t)]P^\lambda(z,t,\gamma^{-1}\xi). \tag{3.8}$$

This expression contains only Möbius transformations acting in the complex plane. Moreover we note that by construction the series (3.5) and (3.8) are term by term identical. So, for example, specifying the c_i as in Eq. (2.8) we get

$$\mathbf{a}^R(\gamma(z,t),\lambda,\xi)[\gamma'(z,t)]=|\gamma'\xi|^\lambda\gamma^{-1\prime}\xi\,\mathbf{a}^R(z,t,\lambda,\gamma^{-1}\xi), \tag{3.9}$$

and the same for $\mathbf{a}^L$ with $\gamma^{-1\prime}\xi$ replaced by $\overline{\gamma^{-1\prime}\xi}$, and the same for $\mathbf{a}^S$ with $\gamma^{-1\prime}\xi$ replaced by $|\gamma^{-1\prime}\xi|$.

The periodization (3.5) and (3.8) of the basis system (2.8) we denote by $\mathbf{a}^{X\Gamma}$, $X=R,L,S$. With Eqs. (3.1) and (3.9) we have

$$\mathbf{a}^{R\Gamma}(z,t,\lambda,\xi)=|\alpha'\xi|^\lambda\alpha'\xi\,\mathbf{a}^{R\Gamma}(z,t,\lambda,\alpha\xi), \tag{3.10}$$

for L and S the $\alpha'\xi$ has to be replaced by $\overline{\alpha'\xi}$ and $|\alpha'\xi|$, respectively.

Finally we discuss the convergence of $\mathbf{a}^{X\Gamma}$. We assume as always, cf. Ref. 3, that Γ is of Schottky or quasi-Fuchsian type, without parabolic cusp singularities, and that ξ is not in the limit set $\Lambda(\Gamma)$. Then the $|z-\gamma\xi|$ in the denominator of $P(z,t,\gamma\xi)$ is uniformly bounded for z and ξ fixed. Using Eq. (3.9) we have

$$|\mathbf{a}^{X\Gamma}|\leqslant\text{const}\sum_{\gamma\in\Gamma}|\gamma'\xi|^{1+\lambda}. \tag{3.11}$$

This series is convergent for $\text{Re}(1+\lambda)>\delta$, δ the Hausdorff dimension of $\Lambda(\Gamma)$, see Ref. 11.

C. The orthogonality relation on F

The $\mathbf{a}^{X\Gamma}(z,t,\lambda=-is,\xi)$, cf. Sec. III B, with $s\in\mathbb{R}$, $\xi\in\cup f_k$ (the free faces of F), constitute a complete orthogonal set on F, provided δ [see below Eq. (3.11)], satisfies $\delta\leqslant 1$. If $\delta>1$ we will have to supplement this set, see Sec. III D. The Hilbert space scalar product for vector fields on F is given by Eq. (2.11) with H^3 replaced by F.

The orthogonality relation on F is easily derived by using the orthogonality relation (2.13) in the covering space:

$$\begin{aligned}\langle\mathbf{a}^{X\Gamma}(\lambda,\xi),\mathbf{a}^{Y\Gamma}(\lambda',\xi')\rangle_F&=\sum_{\gamma\in\Gamma}\int_F\langle\mathbf{a}^X(\gamma(z,t),\lambda,\xi)[\gamma'(z,t)],\mathbf{a}^{Y\Gamma}(z,t,\lambda',\xi')\rangle d\mu_{H^3}(z,t)\\&=\sum_{\gamma\in\Gamma}\int_{\gamma(F)}\langle\mathbf{a}^X(z,t,\lambda,\xi),\mathbf{a}^{Y\Gamma}(z,t,\lambda',\xi')\rangle d\mu_{H^3}\\&=2\pi^3a^3(\tau)R^5s^{-2}\delta(\xi-\xi')\delta(s-s'),\end{aligned} \tag{3.12}$$

where we have put $\lambda=-is$, $\lambda'=-is'$.

In the derivation of Eq. (3.12) we used at first the Γ-invariance of $d\mu_{H^3}$, of g^{ij} in the scalar product (2.9), and of $\mathbf{a}^{Y\Gamma}$, cf. Eq. (3.1). Then the integrand gets independent of the γ, and we may use the tiling property, $\cup_{\gamma\in\Gamma}\gamma(F)=H^3$, and replace $\Sigma_{\gamma\in\Gamma}\int_F$ by $\int_{H^3}$. After that we expand $\mathbf{a}^{Y\Gamma}$, using its series representation (3.8). The points ξ and $\gamma(\xi')$ can never meet, because $\xi,\xi'\in\cup f_k$, and the free faces $\cup f_k$ constitute a fundamental domain of Γ on $\mathbb{C}$. Thus only the term $\gamma=id$ in the expansion of $\mathbf{a}^{Y\Gamma}$, the same as in Eq. (2.13), gives a nonzero contribution to Eq. (3.12). Finally, if $\delta>1$, the series (3.8) does not converge absolutely, and the $\mathbf{a}^{X\Gamma}$, $\mathbf{a}^{Y\Gamma}$ at $\lambda=-is$ are then meant in the sense of analytic continuation, see also Sec. III D.

D. The completeness relation on F

Keeping in mind the completeness relation in the covering space H^3, cf. Eq. (2.15), we start with the integral

$$H_{ij}:=\frac{i}{4\pi^3 a^3(\tau)R^5}\int_{\mathrm{Re}(\lambda)=0} d\lambda\, R_{ij}^{\Gamma}(z,t,z',t',\lambda), \tag{3.13}$$

$$R_{ij}^{\Gamma}(z,t,z',t',\lambda):=\lambda^2\int_{\cup f_k} d\xi[a_i^{R\Gamma}(z,t,\lambda,\xi)\overline{a_j^{R\Gamma}(z',t',-\bar{\lambda},\xi)}+R\to L+R\to S]. \tag{3.14}$$

If $\delta<1$ then both series $\mathbf{a}^{X\Gamma}(\lambda)$ and $\mathbf{a}^{X\Gamma}(-\bar{\lambda})$ are absolutely convergent at $\mathrm{Re}(\lambda)=0$, and we can readily expand them, like in the derivation of the orthogonality relation (3.12). For the case $\delta\geqslant 1$ we have to make some analyticity assumptions on the $\mathbf{a}^{X\Gamma}$.

It is well known[11] that the series (3.11) is analytic in the half-plane $\mathrm{Re}(\lambda)>-1$, with the exception of a pole of the first order at $\lambda=\delta-1$. We make the same analyticity assumption on $\mathbf{a}^{S\Gamma}$, in particular that it has a pole of the first order at $\lambda=\delta-1$, which we will verify later. The $\overline{\mathbf{a}^{S\Gamma}(-\bar{\lambda})}$ has then its pole at $\lambda=1-\delta$ with the same residue, apart from a change of sign. We note that the series $\mathbf{a}^{R\Gamma}$, $\mathbf{a}^{L\Gamma}$ have no poles in the region $\mathrm{Re}(\lambda)>-1$ because the pole at $\lambda=\delta-1$ of the series of the absolute values is wiped out by the phases of $\gamma'\xi$. Thus the integrand (3.14) has poles of the first order at $\lambda^+=\delta-1$, and $\lambda^-=1-\delta$, stemming from $\mathbf{a}^{S\Gamma}\overline{\mathbf{a}^{S\Gamma}}$.

At first we shift the path in Eq. (3.13) to the right, $\mathrm{Re}(\lambda)=\lambda^++\epsilon$, then we can expand $\mathbf{a}^{X\Gamma}(\lambda)$. Using the same invariance properties as in the derivation of Eq. (3.12) we get

$$H_{ij}=\frac{i}{4\pi^3 a^3(\tau)R^5}\int_{\mathrm{Re}(\lambda)=\lambda^++\epsilon} d\lambda\,\lambda^2\int_{\mathbf{C}} d\xi[a_i^{R}(z,t,\lambda,\xi)\overline{a_j^{R\Gamma}(z',t',-\bar{\lambda},\xi)}+R\to L$$

$$+R\to S]+\frac{1}{2\pi^2 a^3(\tau)R^5}\,\mathrm{res}_{\lambda=\lambda^+}R_{ij}^{\Gamma}(z,t,z',t',\lambda,\xi). \tag{3.15}$$

In order to expand the $\overline{\mathbf{a}^{X\Gamma}}$ in Eq. (3.15) we have to shift the path of integration to $\mathrm{Re}(\lambda)=\lambda^--\epsilon$, across the pole of $\overline{\mathbf{a}^{S\Gamma}(-\bar{\lambda})}$ at λ^-. Then we can interchange summation and integration in Eq. (3.15). We use now like in Eq. (3.12) the fact that (z,t) and $\gamma(z',t')$ can never come close to each other if both (z,t) and $(z,',t')$ lie in F. So only the term $\gamma=id$ gives a nonzero contribution to the series, and using Eq. (2.15) and $\mathrm{res}_{\lambda=\lambda^+}R^{\gamma}=-\mathrm{res}_{\lambda=\lambda^-}R^{\Gamma}$, we arrive at

$$\int_{\mathbf{R}\times\cup f_k} d\sigma_{H^3}(s,\xi)\,[a_i^{R\Gamma}(z,t,\lambda,\xi)\overline{a_j^{R\Gamma}(z',t',-\bar{\lambda},\xi)}+R\to L+R\to S]$$

$$-\frac{1}{\pi^2 a^3(\tau)R^5}\,\mathrm{res}_{\lambda=\lambda^+}R_{ij}^{\Gamma}(z,t,z',t',\lambda)=g_{ij}\delta_{\mu_{H^3}}(z,t;z',t'), \tag{3.16}$$

with $d\sigma_{H^3}$ and $\delta_{\mu_{H^3}}$ as in Eq. (2.15). In Ref. 2, Eq. (A7), an analytic expression for $\delta_{\mu_{H^3}}$ can be found. Clearly, if $\delta<1$ then the residual part in Eq. (3.16) is zero.

Next we extract the residue of R_{ij}^{Γ} in Eq. (3.14). Formally expanding, we have

$$R_{ij}^{\Gamma}(z,t,z',t',\lambda;h)=\sum_{\gamma\in\Gamma}R_{ik}(z,t,\gamma(z',t'),\lambda;h)\,[\gamma'(z',t')]_j^k, \tag{3.17}$$

with

$$R_{ij}(z,t,z',t',\lambda;h):=\lambda^2(1+L(z,t;z',t'))^{-h}\int_{\mathbf{C}} d\xi[a_i^R(z,t,\lambda,\xi)\overline{a_j^R(z',t',-\bar{\lambda},\xi)}+R\to L+R\to S], \tag{3.18}$$

where we applied the same invariance properties as in the derivation of Eq. (3.12), and $d(\gamma\xi)=|\gamma'\xi|^2 d\xi$. Moreover we have introduced a convergence factor $(1+L)^{-h}$; L is the scalar point-pair invariant defined in Eq. (A16) of Ref. 2, and h a complex parameter, so that the series (3.17) converges. [Absolute convergence we have for $\mathrm{Re}(1\pm\lambda+h)>\delta$, cf. Eq. (3.11).] The limit $h\to 0$ we obtain then by analytic continuation.

Applying Eq. (3.9) we see that Eq. (3.18) is a vectorial point-pair invariant:

$$R_{nm}(\alpha(z,t),\alpha(z',t'),\lambda;h)[\alpha'(z,t)]_i^n[\alpha'(z',t')]_j^m=R_{ij}(z,t,z',t',\lambda;h), \tag{3.19}$$

for every Möbius transformation α acting on H^3. Likewise, if $\alpha,\beta\in\Gamma$, then

$$R_{nm}^\Gamma(\alpha(z,t),\beta(z',t'),\lambda;h)[\alpha'(z,t)]_i^n[\beta'(z',t')]_j^m=R_{ij}^\Gamma(z,t,z',t',\lambda;h). \tag{3.20}$$

Next we show that

$$\int_F d\mu_{H^3}(z',t')R_{ij}^\Gamma(z,t,z',t',\lambda;h)a^{X\Gamma j}(z',t',\mu,\xi)=\Lambda^X(\lambda,\mu;h)a_i^{X\Gamma}(z,t,\mu,\xi). \tag{3.21}$$

At first, $\Lambda^X(\lambda,\mu;h)$ is independent of ξ, which follows from Eqs. (3.10) and (3.20). Furthermore we show that $\Lambda^S(\lambda,\mu;h=0)$ has poles of first order at $\lambda=\pm\mu$, and calculate the residues. As mentioned, the $\Lambda^{R,L}$ have no poles, since the $\mathbf{a}^{R,L}$ have none. Because the $\mathbf{a}^{R,L}$ are eigenfunctions of Eq. (2.3) we have $\Lambda^R=\Lambda^L$, which follows from the formalism of point-pair invariants,[10,12] but that we will not use in the sequel.

Applying again the invariance properties (3.10), (3.20), and some other properties used in the derivation of Eq. (3.12), we have

$$\int_{H^3} d\mu_{H^3}(z',t')R_{ij}(z,t,z',t',\lambda;h)a^{Xj}(z',t',\mu,\xi)=\Lambda^X(\lambda,\mu;h)a_i^X(z,t,\mu,\xi), \tag{3.22}$$

with the same $\Lambda^X(\lambda,\mu;h)$ as in Eq. (3.21). Moreover we may put $\xi=\infty$, and therefore $a_i^X=a(\tau)t^\mu X_i$, with constant vectors X_i, cf. Eq. (2.8).

As in the scalar case,[10,12] the ξ-integration in R_{ij}, cf. Eq. (3.18), can be carried out by using the fact that R_{ij} is a point-pair invariant: choosing α in Eq. (3.19) so that $\alpha(z,t)=(0,t_1)$, $\alpha(z',t')=(0,t_2)$, the integrals in $R_{ij}(0,t_1,0,t_2,\lambda;h)$ are standard integral representations of hypergeometric functions. The $\Lambda^S(\lambda,\mu;h)$ exhibits poles for $h\to 0$,

$$\Lambda^S(\lambda,\mu;h)\sim -\pi^2a^3(\tau)R^5\left[\frac{1}{\lambda-\mu+h}-\frac{1}{\lambda+\mu+h}\right], \tag{3.23}$$

while the $\Lambda^{R,L}$ stay finite for $h\to 0$ and $\lambda\to\pm\mu$. In Eq. (3.14) we assumed that $\mathbf{a}^{S\Gamma}$ has a pole of the first order at $\lambda^\pm=\pm(\delta-1)$, and therefore also the R_{ij}^Γ. Indeed, if we define now

$$\mathbf{u}^{S\Gamma}(z,t):=\lim_{\lambda\to(\delta-1)}(\lambda-(\delta-1))\mathbf{a}^{S\Gamma}(z,t,\lambda,\xi), \tag{3.24}$$

multiply both sides of Eq. (3.21) by $(\mu-(\delta-1))$, and perform the limit $\mu\to(\delta-1)$, we obtain

$$\int_F d\mu_{H^3}(z',t')\mathrm{res}_{\lambda=\pm(\delta-1)}R^{\Gamma}_{ij}(z,t,z',t',\lambda;h=0)u^{ST\,j}(z',t',\mu,\xi)$$

$$=\mathrm{res}_{\lambda=\pm(\delta-1)}\Lambda^S(\lambda,\mu;h=0)u_i^{ST}(z,t). \tag{3.25}$$

Let us assume for the moment that the $\mathbf{u}^{ST}(z,t)$ in Eq. (3.24) is square-integrable, and orthogonal to the continuous spectrum. Then we can write

$$\mathrm{res}_{\lambda=\pm(\delta-1)}R^{\Gamma}_{ij}(z,t,z',t',\lambda)=\mp\pi^2R^5\hat{u}_i^{ST}(z,t)\overline{\hat{u}_j^{ST}(z',t')}, \tag{3.26}$$

with $\mathbf{u}^{ST}$ normalized as $\langle\hat{\mathbf{u}}^{ST},\hat{\mathbf{u}}^{ST}\rangle_F=a^3(\tau)$, so that $\hat{\mathbf{u}}^{ST}$ is independent of $a(\tau)$. If we insert Eq. (3.26) into Eq. (3.16), we obtain the completeness relation on F.

Finally we note an integral representation of the $\mathbf{u}^{ST}(z,t)$. We derive it from

$$\mathbf{u}^{ST}(z,t)=\lim_{\lambda\to(\delta-1)}(\lambda-(\delta-1))a(\tau)\sum_{\gamma\in\Gamma}(0,0,1)[\alpha'_{\gamma\xi_0}(z,t)]P^{\lambda}(z,t,\gamma\xi_0)|\gamma'\xi_0|^{\lambda+1}, \tag{3.27}$$

where we have used Eqs. (2.8), (3.2), and (3.9).

Therefore we have

$$\mathbf{u}^{ST}(z,t)=a(\tau)\int_{\Lambda(\Gamma)}d\mu(\xi)(0,0,1)[\alpha'_{\xi}(z,t)]P^{\delta-1}(z,t,\xi), \tag{3.28}$$

with[11]

$$d\mu(\xi)=\lim_{\lambda\to(\delta-1)}(\lambda-(\delta-1))\sum_{\gamma\in\Gamma}d\xi\,\delta(\xi-\gamma\xi_0)|\gamma'\xi_0|^{\lambda+1}. \tag{3.29}$$

In Eq. (3.27) we replaced the integration over $\mathbb{C}$ by an integration over $\Lambda(\Gamma)$, because $d\mu(\xi)$ is obviously supported only on the limit set. It is the Hausdorff measure on $\Lambda(\Gamma)$, cf. Ref. 11. Because $\Lambda(\Gamma)$ is compact and $d\mu(\xi)$ is bounded, it is easy to estimate from Eq. (3.28) that $\mathbf{u}^{ST}(z,t)$ is square-integrable for $\delta>1$ (but not for $\delta\leqslant1$), and orthogonal to the $\mathbf{a}^{X\Gamma}(\lambda)$, $\mathrm{Re}(\lambda)=0$, $X=R,L,S$. We emphasize once more, that $\mathbf{u}^{ST}(z,t)$ is not an eigenfunction of Eq. (2.3), but orthogonal to the transversal vector fields $\mathbf{a}^{R\Gamma}$, $\mathbf{a}^{L\Gamma}$ on F, provided $\delta>1$. If $\delta\leqslant1$, then $\mathbf{u}^{ST}$ does not appear at all in the completeness relation (3.16).

IV. ELECTROMAGNETIC ENERGY: THE CLASSICAL VERSION

With $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ we have for the energy-momentum tensor

$$T^{\mu\nu}=\frac{c^2}{4\pi^2}\left[F^{\mu}{}_{\lambda}F^{\nu\lambda}-\frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda}g^{\mu\nu}\right]. \tag{4.1}$$

The special form of the RW line element strongly suggests to use the three-dimensional formalism[13] of $\mathbf{E}$ and $\mathbf{B}$ fields on the spacelike slices:

$$T^{00}=\frac{1}{8\pi^2}[\mathbf{E}^2+\mathbf{B}^2]=:\epsilon,\quad T^{0i}=\frac{c}{4\pi^2}(\mathbf{E}\times\mathbf{B})^i=:c^2p^i,$$
$$T^{ik}=\frac{c^2}{4\pi^2}\left[\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)\gamma^{ik}-B^iB^k-E^iE^k\right]=:c^2\sigma^{ik}, \tag{4.2}$$

ϵ, p, σ are the energy, momentum, and stress densities, respectively. With $\gamma_{ij}=a^2(\tau)h^2\delta_{ij}$, cf. the definitions after Eq. (2.2), we denote in this section the three-space metric, with γ its determinant; the scalar products are to be taken with respect to γ_{ij} (e.g., $\mathbf{E}^2=\gamma^{ik}E_iE_k$), the vectorial products are defined as $(\mathbf{a}\times\mathbf{b})^i=\gamma^{-1/2}\epsilon^{ijk}a_jb_k$ by means of the totally antisymmetric tensors $\sqrt{\gamma}\epsilon_{ijk}$ or $\gamma^{-1/2}\epsilon^{ijk}$.

The covariant components of $\mathbf{E}$ are related with $F_{\mu\nu}$ via $E_i=c^{-1}F_{i0}$, the contravariant components of $\mathbf{B}$ via $B^k=\frac{1}{2}\gamma^{-1/2}\epsilon^{kij}F_{ij}$; inversely, $F_{ij}=\gamma^{1/2}\epsilon_{ijk}B^k$. Finally the contravariant components of a rotor are defined as $(\operatorname{rot}\mathbf{a})^i=\frac{1}{2}\gamma^{-1/2}\epsilon^{ijk}(a_{k,j}-a_{j,k})$, and $\operatorname{div}\mathbf{a}=\gamma^{-1/2}\times(\partial/\partial x^i)(\gamma^{1/2}a^i)$.

Maxwell's equations on the three-slices read[13]

$$\operatorname{rot}\mathbf{E}=\frac{-1}{ca(\tau)}\frac{\partial}{\partial\tau}(a(\tau)\mathbf{B}),\quad \operatorname{div}\mathbf{B}=0,\quad \operatorname{rot}\mathbf{B}=\frac{1}{ca(\tau)}\frac{\partial}{\partial\tau}(a(\tau)\mathbf{E}),\quad \operatorname{div}\mathbf{E}=0, \tag{4.3}$$

from which we obtain the differential identities

$$a^{-4}(\tau)\frac{\partial}{\partial\tau}(a^4(\tau)\epsilon)+c^2\operatorname{div}\mathbf{p}=0, \tag{4.4}$$

and

$$a^{-3}(\tau)\frac{\partial}{\partial\tau}(a^3(\tau)p_i)-\frac{\partial\log h}{\partial x^i}\epsilon+\gamma^{-1/2}\frac{\partial(\gamma^{1/2}\sigma_i^k)}{\partial x^k}=0, \tag{4.5}$$

with ϵ, p, and σ as in Eq. (4.2).

For the total energy, $E=\int_F\epsilon\sqrt{\gamma}\,d^3x$, and the total momentum, $\mathbf{P}=\int_F\mathbf{p}\sqrt{\gamma}\,d^3x$, of the field on a three-slice, we have from Eq. (4.4)

$$a(\tau)E(\tau)=\text{const}, \tag{4.6}$$

and from Eq. (4.5)

$$\frac{d}{d\tau}P_i=\int_F\frac{\partial\log h}{\partial x^i}\epsilon\sqrt{\gamma}\,d^3x, \tag{4.7}$$

as the gravitational force acting on ϵ/c^2.

The $\mathbf{a}^{X\Gamma}(z,t,\lambda=-is,\xi)$, $X=R,L$; $s\in\mathbf{R}$, $\xi\in\cup f_k$ defined in Eqs. (2.8), (3.2), and (3.5), constitute a complete set of transversal eigenfunctions for Eqs. (2.3) and (2.4). From Eqs. (2.8) and (3.2) we have $\mathbf{a}^{R\Gamma}(s)=\overline{\mathbf{a}^{L\Gamma}}(-s)$. Thus every solution of Eqs. (2.1) and (2.2) admits a generalized Fourier expansion

$$\mathbf{A}(z,t)=\operatorname{Re}\left\{a^2(\tau)\int_{\mathbf{R}\times\cup f_k}d\sigma_{H^3}(s,\xi)\mathbf{a}^{R\Gamma}(z,t,s,\xi)\left[b^+(s,\xi)e^{-is\varphi(\tau)}+b^-(s,\xi)e^{+is\varphi(\tau)}\right]\right\}, \tag{4.8}$$

with $\varphi(\tau)=\Lambda\int_{\tau_0}^{\tau}a^{-1}(\tau)d\tau$, $d\sigma_{H^3}$ is the spectral measure defined in Eq. (2.14), and $b^\pm$ are complex amplitudes.

From Eq. (4.8), the definitions after Eq. (4.2), and the orthogonality relation (3.12) we have

$$\int_F \mathbf{E}^2 \sqrt{\gamma}\, d^3x = \frac{a^2(\tau)}{4R^2} \int_{\mathbb{R}\times \cup f_i} d\sigma_{H^3}(s,\xi) s^2 |b^+(s,\xi) e^{-is\varphi(\tau)} - b^-(s,\xi) e^{+is\varphi(\tau)}|^2. \quad (4.9)$$

Using Green's formula, we have for the **B** field

$$\int_F \mathbf{B}^2 \sqrt{\gamma}\, d^3x = -\int_F A_j \frac{\partial(\sqrt{\gamma} F^{ij})}{\sqrt{\gamma}\, \partial x^i} \sqrt{\gamma}\, d^3x. \quad (4.10)$$

Therefore λ^2 in Eq. (2.3) must be negative for bound states, which gives a simple positivity argument that transversal bound states separated from the continuous spectrum cannot exist in the case of the electromagnetic field. Inserting Eqs. (4.8) and (2.3) into Eq. (4.10), we calculate for $\int_F \mathbf{B}^2 \sqrt{\gamma}\, d^3x$ the same expression as in Eq. (4.9), but with the minus sign in the absolute value replaced by a plus sign. We have finally

$$E = \frac{1}{8\pi^2} \int_F (\mathbf{E}^2 + \mathbf{B}^2) \sqrt{\gamma}\, d^3x = \frac{a^2(\tau)}{16\pi^2 R^2} \int_{\mathbb{R}\times \cup f_i} d\sigma_{H^3}(s,\xi) s^2 (|b^+|^2 + |b^-|^2), \quad (4.11)$$

from which Eq. (4.6) follows immediately. In the case that the hyperbolic manifold F is just H^3, we have to replace in Eqs. (4.8) and (4.11) $\mathbf{a}^{R\Gamma}$ by $\mathbf{a}^R$, and $\cup f_i$ by $\mathbb{R}^2$.

Remark: In Refs. 8 and 14 we discussed the positivity of the energy of scalar fields, which are coupled to the curvature scalar $\hat{R}$ of the four-manifold by a dimensionless coupling constant ξ, and which satisfy $[\Box - \xi\hat{R} - (mc/h)^2]\psi = 0$. We found that T_{00} is always positive definite if $0 \leqslant \xi \leqslant 1/6$, and $\lambda - 6\xi > 0$. Here λ is the spectral variable that ranges in $[1,\infty]$, and if $\delta > 1$ it admits a discrete bound state value $\lambda_0 \in [0,1[$. However the energy functional may fail to be positive definite in other cases, for example in the scalar analog to electrodynamics, $m=0$, $\xi = 1/6$, and $\delta > 1$. The requirement of positivity of T_{00} imposes then restrictions on ξ, m, and δ, compare our discussion of de Sitter space in this context.[8]

V. THE ENERGY-MOMENTUM TENSOR OF THE QUANTIZED RADIATION FIELD

The energy density of the quantized radiation field on the hyperbolic manifold F is given by

$$E = 2 \int_{\mathbb{R}\times \cup f_k} d\sigma(s,\xi) h\nu(s,\xi) \left[\exp\left(\frac{h\nu(s,\xi)}{kT} \right) - 1 \right]^{-1} =: \int_0^\infty \rho(\nu) d\nu, \quad (5.1)$$

where s, ξ are the spectral variables, cf. Secs. III and IV; $\cup f_k$ denotes the free faces of F, $d\sigma(s,\xi)$ the density of states in the Fourier space, and $\nu(s,\xi)$ the frequency of the eigenmodes. Formula (5.1) is a straightforward generalization of the Planck formula in Minkowski space. For example, in a RW cosmology with Euclidean three-sections $\mathbb{R}^3$, we can easily determine by box quantization[15] the density of states $d\sigma(\mathbf{k}) = (2\pi)^{-3} a^{-3}(\tau) d^3k$, the frequency $\nu(\mathbf{k}) = c|\mathbf{k}|(2\pi a(\tau))^{-1}$ (the spectral variable $\mathbf{k}$ ranges all over $\mathbb{R}^3$), and the spectral energy density $\rho(\nu) d\nu = 8\pi c^{-3} h[\exp(h\nu/kT) - 1]^{-1} \nu^3\, d\nu$.

In RW cosmologies of negative curvature with the three-space topologies indicated at the end of Sec. III B (covering groups), we have from Eq. (4.8)

$$\nu(s) = \frac{1}{2\pi} \Lambda |s| a^{-1}(\tau). \quad (5.2)$$

The density of states in the Fourier space is best determined in the B^3 model, and given by the spectral measure in the scalar completeness relation (A14) of Ref. 2:

$$d\sigma(s,\eta)=\frac{s^2\,ds\,d\Omega(\eta)}{16\pi^3R^5a^3(\tau)},\quad s\in\mathbb{R}, \tag{5.3}$$

$d\Omega(\eta)$ is the solid angle increment on the sphere $|\eta|=R$, and $\eta\in\cup f_k$.

Clearly the spectral measure is not unique, depending on the normalization of the eigenfunctions. That we have chosen the right normalization we can however see, if we perform the limit: curvature radius $R\to\infty$ in Eq. (A14) of Ref. 2. Then Eq. (A14) reduces to the Euclidean completeness relation, and Eq. (5.3) to the Euclidean spectral measure indicated above. In Secs. II–IV we used the H^3 model of hyperbolic geometry, via Eq. (A11) in Ref. 2 we can transcribe Eq. (5.3) as

$$d\sigma(s,\xi)=\frac{s^2\,ds\,d\xi}{4\pi^3R^5a^3(\tau)(1+|\xi|^2/R^2)^2},\quad s\in R,\quad \xi\in\cup f_k. \tag{5.4}$$

Remark: In order to determine $d\sigma(s,\xi)$ in Eq. (5.1) we cannot apply box quantization, namely, cover the manifold F with a hyperbolic lattice whose cells have finite volume, and impose periodic boundary conditions with respect to this lattice, and finally expand the lattice cells. Hyperbolic lattices with compact cells, e.g., hyperbolic Platonic solids, are rigid and cannot be deformed nor scaled in any way.

The eikonal equation of geometric optics reads

$$g^{\mu\nu}k_\mu k_\nu=0,\quad k_\mu=\frac{\partial\psi}{\partial x^\mu}, \tag{5.5}$$

its complete integral (compare the massive case in Ref. 2) is given by

$$\psi(x,\tau)=\alpha c d(x,x_0)\mp\alpha c^2\int_{\tau_0}^{\tau}a^{-1}(\tau)d\tau, \tag{5.6}$$

α, x_0 are integration constants, and d is the hyperbolic distance function in B^3 or likewise in H^3.

If x_0 is a boundary point (η) we have the horospherical eikonal[2]

$$\psi_\pm=\alpha cR\log P(x,\eta)\pm\alpha c^2\int_{\tau_0}^{\tau}a^{-1}(\tau)d\tau. \tag{5.7}$$

Clearly $|\mathbf{k}|^2_{H^3}=\alpha^2c^2a^{-2}(\tau)=k_0^2$. If we identify α with the energy variable s, $\alpha cR=s$, we can write in Eq. (4.8) $P^{-is}e^{\mp i\varphi(\tau)}=e^{-i\psi_\pm}$, the eikonal appears as the phase of the wave functions. Therefore we are allowed to attach to a ray issuing from a boundary point η a four-momentum $p_\mu=\hbar k_\mu$, i.e.,

$$p_\mu(x,\eta,\nu)=\hbar\frac{\partial\psi_\pm(x,\eta,s(\nu))}{\partial x^\mu}, \tag{5.8}$$

so that $c^2p_0^2=h^2\nu^2$, cf. Eq. (5.2).

Thus the energy-momentum tensor of a horospherical flow of photons in the frequency range $d\nu$, stemming from a solid angle $d\Omega(\eta)$, and having a density $P^2(x,\eta)d\Omega(\eta)$ at x is

$$T_{\alpha\beta}(x,\eta,\nu)d\Omega(\eta)d\nu=\frac{2c}{hR^2}p_\alpha(x,\eta,\nu)p_\beta(x,\eta,\nu)\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}P^2(x,\eta)d\Omega(\eta)\nu\, d\nu, \tag{5.9}$$

compare Refs. 16 and 17 and Eq. (3.1) of Ref. 2. We treat here at first the case that the spacelike slices are just the Minkowski hyperboloid, i.e., hyperbolic space B^3.

If we carry out the $d\Omega(\eta)$ integration in Eq. (5.9) we have

$$\int_{|\eta|=R} T^{00}\, d\Omega(\eta)d\nu=:\rho\, d\nu=\frac{8\pi h}{c^3}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}\nu^3\, d\nu, \tag{5.10}$$

$$\int_{|\eta|=R} T_{0i}\, d\Omega(\eta)d\nu=:c^2p_i\, d\nu=0, \tag{5.11}$$

$$\int_{|\eta|=R} T_{ij}\, d\Omega(\eta)d\nu=:c^2\sigma_{ij}\, d\nu=\frac{1}{3}\frac{8\pi h}{c}\frac{4a^2(\tau)\delta_{ij}}{(1-|x|^2/R^2)^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}\nu^3\, d\nu. \tag{5.12}$$

In the calculation of Eqs. (5.10)–(5.12) we used, cf. the Appendix of Ref. 2,

$$\int_{|\eta|=R} d\Omega(\eta)P(x,\eta)P(y,\eta)=\int_{|\eta|=R} d\Omega(\eta)P(T_yx,\eta)$$

$$=2\pi R^2\frac{(1-r)}{\sqrt{r}}\log\frac{1+\sqrt{r}}{1-\sqrt{r}},$$

$$r:=|T_yx|^2/R^2, \tag{5.13}$$

and

$$\left.\frac{\partial r}{\partial x^i}\right|_{x=y}=\left.\frac{\partial r}{\partial y^i}\right|_{x=y}=0, \quad \text{and} \quad \left.\frac{\partial^2 r}{\partial x^i\,\partial y^j}\right|_{x=y}=\frac{-2\delta_{ij}}{R^2(1-|x|^2/R^2)^2}, \tag{5.14}$$

T_xy is defined in Eqs. (A23) and (A24) of Ref. 2.

The spectral energy density in Eq. (5.10) is exactly the same as in the case of Minkowski space; carrying out the $d\nu$ integration in Eq. (5.1) we arrive at $E=(8\pi^5k^4/15h^3c^3)T^4$. In particular if we assume the time behavior of T to be inversely proportional to $a(\tau)$, then the conservation law (4.4) is satisfied, and likewise Eq. (4.5).

Finally, in comoving coordinates, $u^0=c^{-1}$, $u^i=0$, we have

$$T^{\alpha\beta}\, d\nu=c^2[(\rho+p)u^\alpha u^\beta+pg^{\alpha\beta}]d\nu, \tag{5.15}$$

with the radiation pressure $p=\rho/3$, cf. Eqs. (5.10) and (5.12).

Equation (5.9) can easily be generalized when the three-space is a hyperbolic manifold F with covering group Γ. We obtain the energy-momentum tensor $T^\Gamma_{\alpha\beta}$ on F by periodizing the corresponding tensor in B^3, namely, $T_{\alpha\beta}$ in Eq. (5.9),

$$T^\Gamma_{\alpha\beta}(x,\eta,\nu)d\Omega(\eta)d\nu=\sum_{\gamma\in\Gamma}T_{\mu\kappa}(\gamma x,\eta,\nu)[\gamma' x]^\mu_\alpha[\gamma' x]^\kappa_\beta\, d\Omega(\eta)d\nu, \tag{5.16}$$

and

$$T^{\Gamma}_{\alpha\beta}(x,\nu)d\nu=\int_{\cup f_k} T^{\Gamma}_{\alpha\beta}(x,\eta,\nu)d\Omega(\eta)d\nu. \tag{5.17}$$

The series (5.16) is absolutely convergent, see Eq. (3.11), and use $p_i(\gamma x,\eta,\nu)[\gamma' x]^i_k = p_k(x,\gamma^{-1}\eta,\nu)$.

Note that $P^2(\gamma x,\eta)d\Omega(\eta) = P^2(x,\gamma^{-1}\eta)d\Omega(\gamma^{-1}\eta)$, and

$$\int_{\cup f_k} \sum_{\gamma\in\Gamma} P^2(x,\gamma^{-1}\eta)d\Omega(\gamma^{-1}\eta)=\sum_{\gamma\in\Gamma}\int_{\gamma(\cup f_i)} P^2(x,\eta)d\Omega(\eta)=4\pi R^2,$$

which follows easily from the invariance properties of $P(x,\eta)$ given in the Appendix of Ref. 2. Thus we have $T^{\Gamma}_{00}(x,\nu)d\nu = T_{00}(x,\nu)d\nu$, which means that the Planck formula for the spectral energy density remains unaltered, which is not surprising, because the Maxwell equations have the same spectrum on F and H^3. It is also easy to see that $T^{\Gamma}_{i0}(x,\nu)=0$. We use $p_i(x,\eta,\nu) = [h\nu Ra(\tau)/c][\partial \log P(x,\eta)/\partial x^i]$, and $[\gamma' x]^{-1i}_k[\partial P(\gamma x,\eta)/\partial x^i] = (\partial P/\partial x^k)|_{\gamma(x)}$. The $\partial/\partial x^k$ we can place in front of the sum (5.16), and then we use similar arguments as above.

To calculate $T^{\Gamma}_{ik}(x,\nu)$ we note that

$$\frac{\partial^2}{\partial x^i\,\partial y^j}\sum_{\gamma\in\Gamma}\int_{\cup f_k} P(\gamma x,\eta)P(\gamma y,\eta)d\Omega(\eta)\Big|_{x=y} = \frac{\partial^2}{\partial x^i \partial y^j}\int_{|\eta|=R} P(x,\eta)P(y,\eta)d\Omega(\eta)\Big|_{x=y}, \tag{5.18}$$

and we obtain finally

$$T^{\Gamma}_{\alpha\beta}(x,\nu)d\nu = T_{\alpha\beta}(x,\nu)d\nu, \tag{5.19}$$

which means that the energy-momentum tensor—if averaged over all directions η—is the same on F and H^3.

Now, what is the meaning of formulas (5.9) and (5.16)? Let us place at first the point x on the origin of the Poincaré ball B^3. Then we have $P^2(x=0,\eta)=1$, and $T_{\mu 0}(x=0,\eta,\nu)d\Omega(\eta)d\nu$ is just the part of the energy momentum of the radiation field at $x=0$ that is provided by the photon flow through the solid angle $d\Omega(\eta)$ in the frequency range $d\nu$. Because $T_{00}(x=0)$ does not depend on η, the incoming flow at $x=0$ is perfectly isotropic. For an arbitrary point x in B^3 we choose a Möbius transformation $y\rightarrow T_x y$, so that $T_x x=0$, cf. the Appendix of Ref. 2. Then we have $P^2(x,\eta)d\Omega(\eta) = P^2(T_x x, T_x\eta)d\Omega(T_x\eta) = d\Omega(T_x\eta)$. Thus the factor $P^2(x,\eta)$ takes care that the photon density that arrives at x is independent of the direction η, as is the case for $x=0$.

For an arbitrary hyperbolic manifold F with covering group Γ the isotropy is likewise strictly satisfied: Let x be a point in the fundamental polyhedron F and be $\eta\in\cup f_k$. Consider the horospherical flow bundle issuing from η. How many of these rays will end in x? If $\Gamma=id$, then clearly exactly one. For a generic Γ consider the images γx, $\gamma\in\Gamma$. The trajectories in the manifold, issuing from η and ending in x are exactly the canonical projections of the B^3 geodesics issuing from η and ending in the orbit points γx. Equivalently, another set of covering geodesics for them are the geodesics issuing from $\gamma^{-1}\eta$, $\gamma\in\Gamma$ and ending in x.

Let $p_\mu(x)$ be a given momentum at x. How many trajectories issuing from $\cup f_k$ will end in x, having this momentum? The answer is, as in the topologically trivial case, exactly one for almost all $p_\mu(x)$: Consider the unique B^3 geodesic ending in x with the momentum $p_\mu(x)$. Its initial point η_0 on the boundary S_∞ of B^3 will almost certainly lie in a Γ image, $\gamma(\cup f_k)$, since the limit set $\Lambda(\Gamma)$ has always zero measure (spherical Riemann–Lebesgue). There is then exactly one $\gamma_0^{-1} \in \Gamma$ that carries η_0 into $\cup f_k$. So there is exactly one horospherical bundle, namely, that issuing from $\gamma_0^{-1}(\eta_0)$, whose projection has a ray with $p_\mu(x)$. Because $\gamma^{-1}(\eta_0)$, $\gamma \in \Gamma$, labels all covering trajectories there is exactly one such ray.

The rays whose covering rays have their initial points in $\Lambda(\Gamma)$ do not produce transversal bound states, cf. Sec. III, contrary to scalar fields,[1,2] and therefore they do not effect the spectrum and in particular the isotropy of the radiation field. It would be interesting, and I think also feasible to do a similar analysis for spinor fields.

Finally, we make some comments on the recently discovered large-scale anisotropies[18] in the temperature of the cosmic microwave radiation, which strongly suggests to relax the condition of homogeneity and isotropy. We have just seen that a RW line element always leads to a perfect Planck distribution, even if we impose a topology that breaks the six-dimensional continuous symmetry (in our case of the Lorentz group acting on H^3). On the other hand the multiply connectivity of the infinite three-space provides a natural way to extend RW line elements. Anisotropies on a global level are generated—without giving up the constant sectional curvature $-a^{-2}(\tau)R^{-2}$ of the three-space—by means of a time variation (a generic variation, apart from the variation of the expansion factor) of the metric in the deformation space[7,8] of the three-manifold. Let us sketch here how this comes about.

The fluctuations in the temperature seem to be modest after all, and a possibility to describe it is to take a small and adiabatic perturbation[19] $\tilde{g}_{ij} := g_{ij} + \epsilon h_{ij}(x,\tau)$ of the space part of the RW metric $g_{\mu\nu}$. A solution $\tilde{\psi}$ of the eikonal equation (5.5) in the metric $\tilde{g}_{\mu\nu}$ (with $\tilde{g}_{\mu 0} := g_{\mu 0}$) is then found by expanding (ϵ) around the solution ψ in Eq. (5.7),

$$\tilde{\psi} = \psi + \epsilon \nu a(\tau) \chi(x,\eta,\tau). \tag{5.20}$$

The χ satisfies the linear equation

$$\nu a(\tau) g^{\mu\nu} \frac{\partial \psi}{\partial x^\mu} \frac{\partial \chi}{\partial x^\nu} = -\frac{1}{2} h^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j}. \tag{5.21}$$

Clearly,

$$\tilde{p}_\mu(x,\eta,\tau) = p_\mu(x,\eta,\tau) + \epsilon \hbar \nu a(\tau) \frac{\partial \chi(x,\eta,\tau)}{\partial x^\mu}, \tag{5.22}$$

in particular,

$$\frac{\tilde{p}_0}{h} =: \tilde{\nu} = \nu \left(1 + \frac{\epsilon a(\tau)}{2\pi} \frac{\partial \chi}{\partial \tau} \right). \tag{5.23}$$

The energy-momentum tensor $\tilde{T}_{\alpha\beta}$ in the perturbed metric $\tilde{g}_{\alpha\beta}$ is then given asymptotically by Eq. (5.9), the temperature T replaced by

$$\tilde{T}(x,\eta,\tau) = T \left(1 - \frac{\epsilon a(\tau)}{2\pi} \frac{\partial \chi(x,\eta,\tau)}{\partial \tau} \right). \tag{5.24}$$

To calculate χ in Eq. (5.21) it is necessary to make some choice for the h^{ij}, some global condition on the perturbed $\tilde{g}^{ij}$. It is quite natural to require that the perturbed metric $\tilde{g}_{ij}$ of the three-space shall be of constant sectional curvature $-a^{-2}(\tau)R^{-2}$ as is g_{ij}, cf. Sec. I, and Ref. 3.

Clearly the h^{ij} on the manifold F must be automorphic, $h^{ij}(x)=[\gamma' x]_n^{-1i}[\gamma' x]_m^{-1j}h^{nm}(\gamma x)$, with respect to Γ, compare Eq. (3.1). So we can require for χ the same transformation rule as for ψ, namely, $\chi(\gamma x,\eta,\tau)=\chi(x,\gamma^{-1}\eta,\tau)+f(\eta)$, f being independent of (x,τ). Finally, if we integrate $\widetilde{T}^{\Gamma}{}_{\mu\nu}$, the periodized $T_{\mu\nu}$, over $\cup f_k$ we arrive at

$$\widetilde{T}^{\Gamma}{}_{\alpha\beta}(x,\nu)d\nu \sim \frac{2c}{hR^2}\int_{|\eta|=R} p_\alpha p_\beta(x,\eta,\tau,\nu)\left[\exp\left(\frac{h\nu}{k\widetilde{T}(x,\eta,\tau)}\right)-1\right]^{-1} P^2(x,\eta)d\Omega(\eta)\nu\, d\nu, \tag{5.25}$$

compare Eqs. (5.16)–(5.18).

Then there arises the question how to construct explicitly symmetric automorphic tensor fields h^{ij}, so that $g^{ij}+\epsilon h^{ij}$ gives curvature $-a^{-2}(\tau)R^{-2}$. It is easy to see that the dimension of this deformation space is finite-dimensional, depending on the topology. The construction of such fields is closely related to the construction of quasiconformal deformations[5,8] of Γ. Let F be the fundamental polyhedron, and γ_i the generators of Γ that provide the face identification. Let us assume now that the γ_i are time-dependent, but so that the relations among them (if there are any) are preserved. Accordingly, the fundamental polyhedron $F(\tau)$ varies too. If $F(\tau_0)$ and $F(\tau_1)$ are two polyhedra that are not congruent, (i.e., cannot be mapped onto each other by a Möbius transformation), then the Poincaré metric induced from the covering space H^3 turns them into two globally nonisometric manifolds—but both have the same constant sectional curvature.

That is not yet that what we need, because the transformations $x\to\gamma_i x$ do not leave the RW line element invariant, since γ_i is time-dependent. However there exists a quasiconformal diffeomorphism q_τ of H^3 onto itself, so that $q_\tau(F(\tau))=F(\tau_0)$ and $q_\tau\Gamma(\tau)q_\tau^{-1}=\Gamma(\tau_0)$, compare Sec. 4 of Ref. 8, where such deformations have been constructed on the boundary of H^3. With this q_τ we can transport the metric of $F(\tau)$ onto $F(\tau_0)$, i.e., q_τ applied to the Poincaré metric on H^3 gives a tensor field $\widetilde{g}_{ij}(x,\tau) := [q_\tau^{-1\prime}x]_i^m[q_\tau^{-1\prime}x]_j^n g_{mn}(q_\tau^{-1}(x),\tau)$ that is automorphic with respect to $\Gamma(\tau_0)$, $\widetilde{g}_{ij}(x,\tau) = [\gamma'_{\tau_0}x]_i^m[\gamma'_{\tau_0}x]_j^n\widetilde{g}_{mn}(\gamma_{\tau_0}(x),\tau)$, for all $\gamma_{\tau_0}\in\Gamma(\tau_0)$. The metric $\widetilde{g}_{ij}$ on $F(\tau_0)$ is time-dependent, but both $F(\tau_0)$ and $\Gamma(\tau_0)$ are time-independent, and thus the extended RW line element $ds^2:=-c^2\,d\tau^2+\widetilde{g}_{ij}(x,\tau)dx^i\,dx^j$ is invariant with respect to $\Gamma(\tau_0)$.

VI. CONCLUSION AND OUTLOOK

In this conclusion we reflect a little about wave optics, ray optics, and quantization in the cosmological models treated here. In Secs. II and III we realized that an electromagnetic wave traveling through space–time is a purely geometric–topological object, determined only by the RW line element and the topology of the space–time manifold. (The topology is imprinted on the solutions by imposing periodic boundary conditions on the fundamental polyhedron, i.e., periodicity with respect to the covering group Γ.) In the Maxwell equations there are no additional adjustable parameters involved, in the metric there are two, the speed of light and the curvature radius R. In simply connected spaces like the Minkowski hyperboloid there does not exist a natural length unit, in multiply connected spaces we may take for R the diameter of the domain of chaoticity,[3,7] $C(\Lambda)\setminus\Gamma$, in the three-space manifold.

We constructed the spectral resolution of Maxwell's equations in terms of a complete set of elementary waves generated at some point (η or ξ in Secs. II and III) at infinity of the three-space manifold. These elementary waves are infinitely extended objects and have no direct physical meaning, but superposing them one obtains wave packets of a finite size and one can attach to them an energy as indicated in Sec. IV. This energy is in turn a purely geometric number, cf. Eq. (4.11). Moreover these elementary waves match nicely with geometric ray optics in the cosmological background under consideration. That comes as follows.

Though the eikonal equation may be regarded as the massless limit of the Hamilton–Jacobi equation, it turns out that ray optics gives a rather pale image of classical mechanics. That is so because one cannot define the concept of momentum in the massless limit without introducing a new fundamental constant. In fact it would not be worthwhile to consider ray optics here at all, were it not that in these cosmologies appropriate bundles of such rays, which are by the way unstable and expanding, are just the orthogonal trajectories to the wave fronts of the elementary waves, cf. Sec. V, and Ref. 2. This clearly suggests to define the momentum proportional to the wave vector k_μ, and Planck's constant and Einstein's relation is that what we need to attach to the rays a momentum $\hbar k_\mu$. Having identified the photon momenta with the horospherical wave vectors, it is a straightforward exercise to construct with them the energy-momentum tensor on the three-space manifold, cf. Eqs. (5.9) and (5.16).

There are three ingredients in Planck's radiation formula (5.1), the density of eigenmodes, their frequency, and finally the distribution function of the eigenmodes. Quantum mechanics enters two places, via the Einstein relation, attaching to a frequency an energy, and perhaps in the choice of the distribution function.[15] A derivation of this formula from 'first principles' I do not know. But I would like to mention here that the popular interpretation of this formula, namely, representing the field as harmonic oscillators hanging on a lattice in Fourier space cannot be retained, because a corresponding dual hyperbolic lattice in the multiply connected three-space can never exist. Hyperbolic lattices with finite cells are rigid and cannot be expanded into the thermodynamic limit.

A possibility to derive Planck's formula in this context is perhaps to consider in real three-space horospherical bundles of rays. As mentioned the wave fronts are always orthogonal to them, and the classical (Lyapunov) instability of the rays makes these bundles to statistical objects. One could try to do statistics with them instead of the energy of oscillators in momentum space.

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