Cosmological *CP* violation

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Spinor fields are studied in infinite, topologically multiply connected Robertson–Walker cosmologies. Unitary spinor representations for the discrete covering groups of the spacelike slices are constructed. The spectral resolution of Dirac's equation is given in terms of horospherical elementary waves, on which the treatment of spin and energy is based in these cosmologies. The meaning of the energy and the particle–antiparticle concept is explained in the context of this varying cosmic background. Discrete symmetries, in particular inversions of the multiply connected spacelike slices, are studied. The violation of the unitarity of the parity operator, due to self-interference of *P*-reflected wave packets, is discussed. The violation of the *CP* and *CPT* invariance—already on the level of the free Dirac equation on this cosmological background—is pointed out.

I. INTRODUCTION

Cosmology—so far as the global structure of the universe is concerned—has always been somewhat at odds with the basic criterion of verifiability, that we rightly impose, since Galilei's time, onto a physical discipline.

This has lead to a very cautious attitude of many eminent physicists toward cosmological modeling. So, for example, the idea that the universe is infinite has been rejected on the grounds that we will not be able to look at infinity, and to verify what is happening there.[1] Other quite attractive arguments, like the atomicity of matter,[2] or Mach's principle,[3] were put forward to plead for the closure of space. A noticeable exception is Ref. 4. I think it is also fair to say, that closed universes are handier for heuristic reasoning.[5,6]

Although we are not able to look at infinity, there may be nevertheless the possibility to test infinite cosmological models, provided that local, microscopic phenomena are influenced by the global topological and metrical structure of the universe. It may be also appropriate to mention here that over the centuries most natural philosophers attached to the concept "Universe" the attribute "infinite," and some kind of nontrivial evolution toward a nontrivial end.

Consequently we should abandon the reasoning "we do not know the global structure, and therefore we assume the simplest possible." Instead we have to try different topological and metrical scenarios, and to find traces of them in the quantum fields, and then we have to see what makes the difference.

In practice this means that we should leave in our arguments the details of the global structure as long as possible unspecified, for example, the specific structure of the generators of the covering group of the spacelike slices, similarly as is usually done with the expansion factor in the topologically trivial models.

We must keep in mind that the ultimate question is not so much what *is* the global structure of the universe, but rather *how does it evolve*. In fact, our basic assumptions,[7] that space is infinite, that it is multiply connected, and that it has constant negative curvature, match well in this respect. They make a dynamic evolution of the metric[8,9] (by global deformations of the three-space manifold which do not change the curvature, but lead to nonisometric spacelike slices), and ultimately also transitions from one topology to another (there is some similarity with Wheeler's superspace[3]) possible. In closed universes with compact hyperbolic manifolds as spacelike slices such deformations are not possible (Mostow rigidity theorem[10]), without introducing distortions of the curvature. (However, a closed multiply connected universe with Euclidean three-sections may well undergo global deformations.[9])

Even if we do not know the actual laws of the cosmic evolution yet, we can nevertheless make some preliminary conclusions.[9] Open Robertson–Walker (RW) universes with multiply connected hyperbolic spacelike slices have a finite chaotic center,[7] that in my opinion is the reason for the remarkable though apparently imperfect uniformity of the galactic background. Global metrical deformations lead to annihilation and production processes in wave fields,[11] which are otherwise freely propagating in space. In particular such deformations can lead to particle production processes in neutrino fields, which cannot be generated by a mere variation of the expansion factor, cf. Example 2 in Sec. VI, where positive and negative frequencies stay perfectly well defined.[12] The angular anisotropy in the temperature of the microwave background is likewise a natural consequence of a dynamic evolution by metrical deformations.[8]

In this article we report on another striking topological phenomenon: parity, the space-reflection symmetry, is violated on the most fundamental level, namely, in the free Dirac equation on a multiply connected cosmological background. There is substantial evidence that *C*, *P*, and *T* are not generic symmetries on the microscopic level. Quantum field theory can cope easily with that, but to describe it one has to add deliberately symmetry breaking interaction terms to the Lagrangians. Concerning neutrinos, if we describe them by four-component spinors, we must exclude without *a priori* reason half of the possible solutions of the Dirac equation. Two-component equations, cf. Refs. 13 and 14, constitute a possible adaptation to this situation, but do not really provide a reason why there are no right-handed neutrinos and left-handed antineutrinos. Why should left-handed neutrinos and right-handed antineutrinos be preferred to their counterparts, as long as there is no natural mechanism to break the space reflection or charge conjugation symmetry?

Now, the *T* symmetry is violated because of the expansion of the space. That is actually not a surprising thing, and occurs in practice also in classical probabilistic systems. In order to render such classical systems time symmetric one would have to prepare initial and end value conditions with infinite precision. In fact, geodesic motion in this context is a good example for that.[15]

The violation of the space-inversion symmetry has now really no classical counterpart. It is an interference phenomenon, that stems from the fact that in a multiply connected space a *P*-reflected wave packet can overlap with itself. Finally, *C* is still a good symmetry of the free Dirac equation, but the unitarity of *CP* and noticeably *CPT* is violated. (For the definition of these operations in hyperbolic space see Sec. V.) Clearly one is tempted to speculate if this can lead, if combined with particle annihilation–creation processes (which occur likewise in the free Dirac equation, cf. Example 3 in Sec. VI), to a dynamic generation of the baryon asymmetry in the universe. But it is outside the scope of this article to give quantitative evidence, and we will also not discuss how this topological symmetry breaking relates to current particle phenomenology. In particular we will not address the important question why *CP* violation appears to be so weak that it has up to now only been confirmed in kaon systems. Our main objective is to show, that *P* is not a natural symmetry on the microscopic level, as soon as one passes from the classical point-particle concept to wave equations in the context of a multiply connected cosmology.

The article is organized as follows. In Sec. II we study the continuous symmetries of the Dirac equation in hyperbolic space H^3. They are generated by the orientation preserving invariance group $SL(2,\mathbb{C})/\{\pm \mathrm{id}\}$ of the hyperbolic metric. We construct the spinor representation of $SL(2,\mathbb{C})$ in Poincaré's half-space model of hyperbolic geometry.

In Sec. III we give the spectral resolution of Dirac's equation in an open and simply connected RW cosmology with arbitrary expansion factor. This has been done in Ref. 16 for closed RW cosmologies. Evidently, if one replaces in this article the spherical functions by pseudospherical ones, one can get also the spectral resolution for the open models. On the other hand we present the spectral theory in a way that can be easily adapted to multiply connected cosmologies, which do not admit continuous symmetries. We give the spectral resolution in the Poincaré half-space in terms of horospherical elementary waves.

In Sec. IV we give a description of spin in hyperbolic spaces. Spin just appears as a vector field on the wave fronts of the elementary waves. In Sec. V we construct the (anti-)unitary representations of the *C-P-T* symmetries in H^3, and discuss their meaning in simply connected RW cosmologies.

The universal covering space of the multiply connected spacelike slices of our RW models is homogeneous, and the semiclassical approximation happens to be exact in it, cf. Sec. VI and Ref. 17 for a more precise statement on that. This phenomenon leads to interesting relations between classical geodesic motion and wave mechanics, cf. Refs. 17 and 18, and in this article we will use it again to clarify the energy concept for spinor fields in (multiply connected) RW cosmologies with negatively curved spacelike slices. In Sec. VI we discuss the meaning of the particle–antiparticle concept and the meaning of the field energy in the context of horospherical waves traveling in the expanding space, and give several examples.

In Sec. VII we discuss spinorial wave fields in multiply connected RW cosmologies, and give explicitly the spectral resolution of the Dirac equation in this context. In Sec. VIII we construct and study space inversions in a multiply connected RW geometry. The discrete symmetries *C*, *P*, and *T* still admit spinor representations that are symmetries of the Dirac equation. However, the unitarity of the *P* operation cannot be retained in the case of a multiply connected space. In Sec. IX we present further discussion, and in the Appendix we summarize our notation and some technical things concerning spinors in hyperbolic spaces.

For the basic concepts of spinor calculus in curved spaces we refer to Refs. 19 and 20. The formalism and the notation concerning covariant spinorial differentiation we have taken over from Ref. 20, which together with Refs. 7 and 17 is a prerequisite, otherwise this article is essentially self-contained. But a certain familiarity with quantum field theory in curved spaces would enhance the understanding of this article, see the standard reviews on this subject,[21,22] where also an extensive list of more recent references on spinor theories in curved spaces can be found. I add here Ref. 23, where the Gordon decomposition of the current is discussed in the context of spatially flat RW cosmologies. I emphasize however that we will not attempt to do second quantization, nor did we in the preceding articles.[7,17,8] What we are studying here is wave mechanics, and how it relates to the topology of the underlying space.

II. TRANSFORMATION THEORY OF THE DIRAC EQUATION IN SIMPLY CONNECTED RW COSMOLOGIES OF NEGATIVE SPATIAL CURVATURE

We study the Dirac equation on $\mathbf{R}^{(+)}\times H^3$, where $\mathbf{R}^{(+)}$ denotes the time (semi-)axis, and H^3 the Minkowski hyperboloid, or equivalently the Poincaré half-space. The metric is defined in Eq. (A1); for the basic definitions and our notation we refer to the Appendix and Refs. 7 and 17.

Dirac's equation reads[19,20]

$$(\gamma^\lambda \nabla_\lambda + \mu)\psi = 0, \tag{2.1}$$

with γ^μ as in Eq. (A7), ∇_λ as in Eq. (A14), and $\mu := mc/\hbar$.

More explicitly, we may write Eq. (2.1) as

$$\left[-c^{-2}\,\tilde{\gamma}_0 \frac{\partial}{\partial x^0} + \frac{x^3}{Ra(\tau)}\,\tilde{\gamma}_i \frac{\partial}{\partial x^i} - \frac{3}{2}\,\tilde{\gamma}_0\, c^{-2}\frac{\dot{a}}{a} - \frac{1}{Ra}\,\tilde{\gamma}_3 + \mu \right]\psi = 0, \tag{2.2}$$

where the $\tilde{\gamma}_\mu$ are the Dirac matrices in Minkowski space, defined in Eq. (A5).

It is quite instructive to square Eq. (2.1) (cf. Ref. 19)

$$(\gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu - \mu^2)\psi = 0 \tag{2.3}$$

and we have

$$\gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu = \Box - \frac{1}{4}\hat{R}\ \mathrm{id}, \quad \Box := \frac{1}{\sqrt{-g}} \nabla_\mu [\sqrt{-g} g^{\mu\nu} \nabla_\nu]. \tag{2.4}$$

$\hat{R}$ is the curvature scalar as defined in Eq. (6) of Ref. 7. Because of the connection coefficients Γ_μ in ∇_μ, cf. Eq. (A14), the operator $\Box$ does *not* split into four uncoupled Klein–Gordon equations, one for each component of ψ, as is the case in Minkowski space.

The continuity equation

$$j^\mu{}_{;\mu} = 0, \quad j^\mu := \bar{\psi}\gamma^\mu \psi \tag{2.5}$$

we obtain by combining Eq. (2.1) with its adjoint

$$(\nabla_\nu \bar{\psi})\gamma^\nu - \mu\bar{\psi} = 0, \tag{2.6}$$

where the covariant derivative of $\bar{\psi}$ is defined in Eq. (A14).

We define a scalar product

$$\langle \psi_1, \psi_2 \rangle := \int_\Sigma \bar{\psi}_2 \gamma_\mu \psi_1 \, d\Sigma^\mu = \int_{H^3} \psi_2^\dagger \psi_1 \, dV_{H^3} \tag{2.7}$$

here Σ is an arbitrary spacelike hypersurface, which we choose to be H^3, and dV_{H^3} is the volume element of γ_{ij}, the three-space metric, as defined after Eq. (A3).

Whenever it is possible to disentangle positive and negative frequencies, for example, in a period in which $a(\tau)$ is constant, we can define the energy of a wave field as

$$E(\psi) := \langle \psi, \psi \rangle^{-1} \int_\Sigma T_{0\mu}(\psi) d\Sigma^\mu = \langle \psi, \psi \rangle^{-1} \int_{H^3} T_{00}(\psi) dV_{H^3}, \tag{2.8}$$

with the energy-momentum tensor[19,20]

$$T_{\mu\nu} := \tfrac{1}{2}(\Theta_{\mu\nu} + \Theta_{\nu\mu}), \quad \Theta_{\mu\nu} := \frac{\hbar i}{2} [\bar{\psi}\gamma_\mu \nabla_\nu \psi - (\nabla_\mu \bar{\psi})\gamma_\nu \psi]. \tag{2.9}$$

The meaning of Eq. (2.8) will be extensively discussed at the end of Sec. VI.

After these introductory preparations we come now to the main part of this section, namely, the transformation properties of the solutions of Eq. (2.1) with respect to the invariance group of H^3.

The orientation preserving symmetry group on H^3 is $SL(2,\mathbb{C})/\{\pm \mathrm{id}\}$, acting as Möbius transformations,[24] see also Eq. (2.15). Let $x^{i'} = h^i(x)$ be a Möbius transformation on H^3, which we extend trivially to $\mathbf{R}^{(+)} \times H^3$ by $h^0 : x^{0'} = x^0$. We have to find a transformation rule

$$\psi(x) \to S_h(x)\psi(h(x)) \tag{2.10}$$

for a spinor ψ satisfying Eq. (2.1), with a four by four matrix $S_h(x)$, so that $S_h(x)\psi(h(x))$ satisfies again Eq. (2.1).

Equation (2.1) reads, if expressed in x'

$$\left[\gamma^{\lambda}(h^{-1}(x'))f^{\kappa}_{\lambda}(x')\frac{\partial}{\partial x'^{\kappa}}-\Gamma_{\lambda}(h^{-1}(x'))+\mu\right]S_h(h^{-1}(x'))\psi(x')=0, \tag{2.11}$$

with $f^{\kappa}_{\lambda}(x'):=\partial h^{\kappa}(x)/\partial x^{\lambda}|_{x=h^{-1}(x')}$. The Γ_{μ} are the connection symbols in ∇_{μ}, cf. Eq. (A12). Multiplying Eq. (2.11) on the left by $S_h^{-1}(h^{-1}(x'))$, we get immediately the conditions that S_h has to satisfy in order that Eq. (2.11) is fulfilled,

$$\gamma^{\lambda}(x)\frac{\partial h^{\kappa}(x)}{\partial x^{\lambda}}=S_h(x)\gamma^{\kappa}(h(x))S_h^{-1}(x) \tag{2.12}$$

and

$$S_h^{-1}(x)\gamma^{\lambda}(x)\Gamma_{\lambda}(x)S_h(x)-S_h^{-1}(x)\gamma^{\lambda}(x)\frac{\partial S_h(x)}{\partial x^{\lambda}}=\gamma^{\lambda}(h(x))\Gamma_{\lambda}(h(x)). \tag{2.13}$$

Equation (2.12) we encounter in Minkowski space, when h is a Lorentz transformation.[14] We construct now matrices that satisfy Eqs. (2.12) and (2.13). Note that h does not mix space and time coordinates. The special form of the Dirac matrices in Eqs. (A7) and (A5) suggests to try the ansatz

$$S_h:=\begin{pmatrix}\hat{S}_h & 0\\ 0 & \hat{S}_h\end{pmatrix}, \tag{2.14}$$

with two by two matrices $\hat{S}_h$.

The invariant action of a SL(2,C) matrix $h:=\binom{ab}{cd}$ in H^3 is given by the Möbius transformation

$$h:\ (z,t)\rightarrow\left[\frac{(az+b)\overline{(cz+d)}+a\bar{c}t^2}{|cz+d|^2+|c|^2t^2},\frac{t}{|cz+d|^2+|c|^2t^2}\right] \tag{2.15}$$

see Eq. (A2) for the definition of (z,t). Note that h and $-h$ give the same transformation. So the Möbius transformations (2.15) provide also a representation of $SL(2,C)/\{\pm id\}$. We use the same symbol for matrices and Möbius transformations, whenever no confusion can arise.

To find $\hat{S}_h$ we decompose the matrix h into

$$h=T_{a/c}\circ R\circ H_c\circ T_{d/c},\quad T_a:=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix},\quad R:=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad H_c:=\begin{pmatrix}c & 0\\ 0 & 1/c\end{pmatrix}. \tag{2.16}$$

In order that Eq. (2.10) is a representation of SL(2,C), we must have, for $g,h\in SL(2,C)$

$$S_{g\circ h}(x)=S_h(x)S_g(h(x)) \tag{2.17}$$

and therefore it is enough to solve Eq. (2.12) for $h=T,R,H$ as in Eq. (2.16), and to compose according to Eq. (2.17). In this way we calculate readily

$$\hat{S}_h(z,t)=\frac{1}{\sqrt{|cz+d|^2+|c|^2t^2}}\begin{pmatrix}\overline{cz+d} & ct\\ -\bar{c}t & cz+d\end{pmatrix}. \tag{2.18}$$

As indicated in the Appendix we switch freely between (z,t) and x^i, whenever it is convenient.

Equations (2.10), (2.14), and (2.18) define a representation of SL(2,C); one checks easily that Eq. (2.12) and in particular Eq. (2.13) holds true. It is evident that this representation is unitary with respect to the scalar product (2.7). For the adjoint spinor the transformation rule corresponding to Eq. (2.10) is $\bar{\psi}(x) \to \bar{\psi}(h(x))S_h^{-1}(x)$.

If we change the sign of the matrix h in Eqs. (2.15) and (2.18), then S_h changes likewise the sign. Therefore Eq. (2.10) is only a ray representation of SL(2,C)/{±id}, very similar to the situation in Minkowski space. This will force us later to introduce for discrete subgroups of SL(2,C)/{±id} a character system in order to obtain via Eqs. (2.10), (2.14), and (2.18) unambiguous representations of them, cf. Sec. VII.

III. SPECTRAL THEORY OF THE DIRAC EQUATION IN HYPERBOLIC SPACE H^3

To find the spectral resolution of Eqs. (2.2), (2.7), we use techniques quite similar to those developed for the electromagnetic field in Ref. 8. At first we try to find solutions whose horospherical wave fronts are parallel to the complex plane, the boundary at infinity of H^3. These elementary waves do not depend on the $z(=x^1+ix^2)$ coordinates in H^3. Then we apply certain transformations (2.10) to these solutions, to generate a complete set. In Eq. (2.2) we try the separation ansätze

$$\psi=t^{1+is}(\varphi_1(s,\tau),0,\varphi_3(s,\tau),0)^t, \quad \psi=t^{1+is}(0,\varphi_2(s,\tau),0,\varphi_4(s,\tau))^t. \tag{3.1}$$

We write here t instead of x^3; for convenience we write spinors as transposed line vectors. The s is a complex separation (spectral) parameter, it will later turn out that it is enough to take s real. We obtain from the first ansatz

$$D_\tau^- \varphi_1(s,\tau)-scR^{-1}\varphi_3(s,\tau)=0, \quad D_\tau^+\varphi_3(s,\tau)-scR^{-1}\varphi_1(s,\tau)=0, \tag{3.2}$$

with

$$D_t^\pm :=ia(\tau)\frac{d}{d\tau}+\frac{3}{2}i\dot{a}(\tau)\pm a(\tau)\mu c. \tag{3.3}$$

For the second ansatz in Eq. (3.1) we get the same Eqs. (3.2) with the minus sign in front of scR^{-1} changed into plus, and the subscripts (1,3) replaced by (2,4). We treat from now on only Eqs. (3.2), and give later the results for the second ansatz.

The system (3.2) is obviously equivalent to

$$D_\tau^+ D_\tau^- \varphi_1 - s^2c^2R^{-2}\varphi_1=0, \tag{3.4}$$

$$\varphi_3=s^{-1}c^{-1}RD_\tau^-\varphi_1; \tag{3.5}$$

note that

$$\overline{D_\tau^+}=-D_\tau^-, \quad D_\tau^+D_\tau^-=\overline{D_\tau^-D_\tau^+}. \tag{3.6}$$

In Eqs. (3.4) and (3.5) we may interchange simultaneously φ_1 and φ_3, and D_τ^- and D_τ^+. From Eq. (3.4) it is clear that φ_1 depends only on s^2 (rather than s), and the same holds true for φ_3. We write from now on $\varphi_1(s^2,\tau)$, $\varphi_3(s^2,\tau)$.

We list some straightforward properties of solutions $\varphi(s^2,\tau)$ of Eq. (3.4). We replace in Eq. (3.4) s by $\bar{s}$, and denote a solution of this equation by $\varphi(\bar{s}^2,\tau)$. Then $\tilde{\varphi}(s^2,\tau)$: $= D_\tau^+\overline{\varphi(\bar{s}^2,\tau)}$ is a solution of Eq. (3.4). We define for two solutions $\varphi(s^2,\tau)$ and $\varphi(\bar{s}^2,\tau)$

$$W:=-ia(\tau)\left[\overline{D_\tau^+\varphi(\bar{s}^2,\tau)}\frac{d}{d\tau}\varphi(s^2,\tau)-\varphi(s^2,\tau)\frac{d}{d\tau}\overline{D_\tau^+\varphi(\bar{s}^2,\tau)}\right], \tag{3.7}$$

which can also be written as

$$W=D_\tau^-\varphi(s^2,\tau)\overline{D_\tau^-\varphi(\bar{s}^2,\tau)}+s^2c^2R^{-2}\varphi(s^2,\tau)\overline{\varphi(\bar{s}^2,\tau)}. \tag{3.8}$$

Moreover

$$\frac{d}{d\tau}(a^3W)=0. \tag{3.9}$$

Therefore, if s is real, $\varphi(s^2,\tau)$ and $\tilde{\varphi}(s^2,\tau):=\overline{D_\tau^+\varphi(s^2,\tau)}$ constitute a fundamental system for Eq. (3.4).

More explicitly, Eq. (3.4) reads

$$\ddot{\varphi}+4\frac{\dot{a}}{a}\dot{\varphi}+\left(\frac{3}{2}\frac{\ddot{a}}{a}+i\frac{\dot{a}}{a}\mu c+\frac{9}{4}\frac{\dot{a}^2}{a^2}+\mu^2c^2+\frac{s^2c^2}{a^2R^2}\right)\varphi=0. \tag{3.10}$$

With these prerequisites we turn back to the ansätze (3.1) and write down the following solutions of Eq. (2.2):

$$\hat{\psi}(s,t,\tau,1,1)=it^{1+is}(scR^{-1}\varphi(s^2,\tau),0,D_\tau^-\varphi(s^2,\tau),0)^t, \tag{3.11}$$

$$\hat{\psi}(s,t,\tau,1,-1)=it^{1+is}(0,scR^{-1}\varphi(s^2,\tau),0,-D_\tau^-\varphi(s^2,\tau))^t, \tag{3.12}$$

$$\hat{\psi}(s,t,\tau,-1,1)=-it^{1+i\bar{s}}(\overline{D_\tau^+\varphi(s^2,\tau)},0,\bar{s}cR^{-1}\overline{\varphi(s^2,\tau)},0)^t, \tag{3.13}$$

$$\hat{\psi}(s,t,\tau,-1,-1)=it^{1+i\bar{s}}(0,\overline{D_\tau^+\varphi(s^2,\tau)},0,-\bar{s}cR^{-1}\overline{\varphi(s^2,\tau)})^t. \tag{3.14}$$

The first discrete index refers to the energy, i.e., to positive/negative frequencies, whenever these concepts have a meaning—generically that is not the case in a space with a time dependent metric, cf. Sec. VI. The second index refers to the spin, cf. Sec. IV.

The $\hat{\psi}$ in Eqs. (3.11)–(3.14) represent waves whose wave fronts, t=const, are planes parallel to the complex plane $\mathbf{C}$, the boundary of H^3. These planes are in fact horospheres[25,17] emanating from the point at infinity of $\mathbf{C}$. To obtain elementary waves that emerge from some finite point ξ of $\mathbf{C}$ (which is of course still at infinity of H^3), we apply to the solutions (3.11)–(3.14) a symmetry transformation as defined in Eqs. (2.10), (2.14), and (2.18). We choose

$$h=\alpha_\xi:=\begin{pmatrix}0 & i\\ i & -i\xi\end{pmatrix}, \tag{3.15}$$

which, if regarded as a Möbius transformation in the complex plane [$t=0$ in Eq. (2.15)], maps ξ into the point at infinity. Equation (2.18) now reads

$$\hat{S}_{\alpha_\xi}(z,t)=\frac{i}{\sqrt{|z-\xi|^2+t^2}}\begin{pmatrix}-\overline{(z-\xi)} & t\\ t & (z-\xi)\end{pmatrix}. \tag{3.16}$$

By means of Eqs. (2.14) and (3.16) we define, restricting ourselves from now on to real s

$$\psi(s,\xi,z,t,\tau,i,k):=S_{\alpha_\xi}(z,t)\hat{\psi}(s,(\alpha_\xi(z,t))_3,\tau,i,k). \tag{3.17}$$

The wave fronts of these elementary waves are horospheres emanating from ξ. More explicitly we have, e.g.,

$$\psi(s,\xi,z,t,\tau,1,1)=\frac{P^{1+is}(z,t;\xi)}{\sqrt{|z-\xi|^2+t^2}}(\overline{(z-\xi)}scR^{-1}\varphi(s^2,\tau),-tscR^{-1}\varphi(s^2,\tau),\overline{(z-\xi)}D_\tau^-\varphi(s^2,\tau),$$
$$-tD_\tau^-\varphi(s^2,\tau))^t, \tag{3.18}$$

with the Poisson kernel

$$P(z,t;\xi):=\frac{Rt}{|z-\xi|^2+t^2}. \tag{3.19}$$

We show now that the ψ in Eq. (3.17), with $s\in\mathbf{R}$, $\xi\in\mathbb{C}$; $i,k=\pm1$, constitute a complete orthogonal set of eigenfunctions of Eqs. (2.1) and (2.2) in H^3, with respect to the scalar product (2.7). To obtain the spectral measure we note that for $s,s'\in\mathbf{R}$

$$\psi^\dagger(s,\xi,z,t,\tau,j,k)\psi(s',\xi',z,t,\tau,m,n)=\delta_{jm}\delta_{kn}[ss'c^2R^{-2}\varphi(s'^2,\tau)\overline{\varphi(s^2,\tau)}$$
$$+D_\tau^-\varphi(s'^2,\tau)\overline{D_\tau^-\varphi(s^2,\tau)}]P^{1-is}(z,t;\xi)P^{1+is'}(z,t;\xi')$$
$$+O(|\xi-\xi'|+|s-s'|). \tag{3.20}$$

Inserting Eq. (3.20) into Eq. (2.7), and applying the scalar orthogonality relation (A4) of Ref. 17, we have

$$\int_{H^3}\psi^\dagger\psi\, dV_{H^3}=\delta_{jm}\delta_{kn}2\pi^3R^5a^3(\tau)W(s^2,\tau)\frac{1}{s^2}\delta(s-s')\delta(\xi-\xi'), \tag{3.21}$$

with $W(s^2,\tau)$ as in Eqs. (3.7)–(3.9). Because s is real we may use in Eq. (3.8) $D_\tau^-\varphi\overline{D_\tau^-\varphi}=D_\tau^+\bar{\varphi}\overline{D_\tau^+\bar{\varphi}}$.

To derive the completeness relation we note

$$\sum_{j,k=\pm1}\psi_\alpha^\dagger(s,\xi,z,t,\tau,j,k)\psi_\beta(s,\xi,z',t',\tau,j,k)$$
$$=\delta_{\alpha\beta}W(s^2,\tau)P^{1-is}(z,t;\xi)P^{1+is}(z',t';\xi)+O(|z-z'|+|t-t'|). \tag{3.22}$$

We can now apply the scalar completeness relation (A5) of Ref. 17. With the spectral measure

$$d\sigma_{H^3}(s,\xi):=\frac{s^2\,ds\,d^2\xi}{4\pi^3R^5a^3(\tau)W(s^2,\tau)} \tag{3.23}$$

we have completeness

$$\int_{\mathbf{R}^3}d\sigma_{H^3}(s,\xi)\sum_{j,k=\pm1}\psi_\alpha^\dagger\psi_\beta=\delta_{\alpha\beta}\delta_{H^3}(z,t;z',t'), \tag{3.24}$$

where δ_{H^3} is the delta function of H^3, cf. Eq. (A7) of Ref. 17, defined with respect to the line element $d\sigma^2$, cf. Eq. (A2).

IV. SPINORS AND SPIN IN HOROSPHERICAL TRIADS

To extract from the ψ in Eq. (3.17) the spin, we endow the spacelike slices with triadic basis fields. To this end we define on H^3 three contravariant vector fields

$$\hat{\mathbf{e}}_1:=a^{-1}(\tau)t(-1,0,0)^t,\quad \hat{\mathbf{e}}_2:=a^{-1}(\tau)t(0,-1,0)^t,\quad \hat{\mathbf{e}}_3:=a^{-1}(\tau)t(0,0,-1)^t. \tag{4.1}$$

We write in the following $\hat{e}_i^k$, the i refers to the triad vector, the k to its components.

Applying to the $\hat{\mathbf{e}}_i$ the coordinate transformation α_ξ defined in Eqs. (3.15) and (2.15), we obtain a new triad $\mathbf{e}_i(z,t;\xi)$,

$$e_i^k(z,t;\xi)=[\alpha_\xi^{-1\prime}(\widetilde{z},\widetilde{t})]_j^k\ \hat{e}_i^j(\widetilde{t})\,|_{(\widetilde{z},\widetilde{t}):=\alpha_\xi(z,t)}, \tag{4.2}$$

where $[\alpha_\xi^{-1\prime}(\widetilde{z},\widetilde{t})]_j^k$ denotes the Jacobian of α_ξ^{-1}.

More explicitly, with $R=\mathrm{Re}(z-\xi)$, $I=\mathrm{Im}(z-\xi)$, and P as in Eq. (3.19), we have

$$\begin{aligned}\mathbf{e}_1&=a^{-1}(\tau)P(z,t;\xi)(-t^2+R^2-I^2,2RI,2tR)^t,\\ \mathbf{e}_2&=a^{-1}(\tau)P(z,t;\xi)(-2RI,t^2+R^2-I^2,-2tI)^t,\\ \mathbf{e}_3&=a^{-1}(\tau)P(z,t;\xi)(2Rt,2It,t^2-R^2-I^2)^t.\end{aligned} \tag{4.3}$$

The $\hat{\mathbf{e}}_i$, if restricted to a horosphere, $t=\text{const}$, constitute three mutually orthonormal [with respect to the metric γ_{ij} defined after Eq. (A3)] vector fields, and so do the $\mathbf{e}_i(z,t;\xi)$ on the horospheres $P(z,t;\xi)=c$. These are Euclidean spheres of radius $1/2c$ and center $(\xi,1/2c)$, with the point $(\xi,0)$ at infinity of H^3 removed.[25] The $\mathbf{e}_1$, $\mathbf{e}_2$ generate the tangent planes, and $\mathbf{e}_3$ is always pointing outwards. From Eqs. (3.18) and (3.19) we see that the wave vector $k^i(z,t;\xi)$ is the γ_{ij} gradient of $-s\log P(z,t;\xi)$, parallel or antiparallel to $\mathbf{e}_3$, and we have $\gamma_{ij}k^ik^j=s^2R^{-2}a^{-2}(\tau)$.

Our discussion of spin is in spirit similar to our discussion of the electromagnetic field in the Coulomb gauge.[8] We employ throughout three-dimensional formalism on the spacelike slices. In fact, the treatment of time just as a parameter (rather than a dimension) is highly encouraged by the form of the RW-line element (A1), which distinguishes time from the space coordinates much more substantially as is the case in Minkowski space. In addition we will later only use coordinate transformations that do not involve time.

We define the spin operator on horospherical wave fronts of the type $t=\text{const}$ as

$$\hat{\Sigma}_k:=\frac{\hbar i}{4}\sqrt{\gamma}\varepsilon_{kij}\gamma^i\gamma^j, \tag{4.4}$$

where $\sqrt{\gamma}\varepsilon_{ijk}$ is the totally antisymmetric tensor of H^3 with respect to γ_{ij}. The γ^i are defined in Eq. (A7). The projection of the vector operator (4.4) onto the triad $\hat{\mathbf{e}}_i$ is just

$$\hat{\Sigma}^P(i):=\hat{\Sigma}_k\ \hat{e}_i^k=\frac{\hbar}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}, \tag{4.5}$$

with the Pauli matrices σ_i as in Eq. (A5).

We construct the spin operator on the horospheres $P(z,t;\xi)=\text{const}$ by means of the coordinate transformation α_ξ

$$\Sigma_k(z,t,\xi):=[\alpha_\xi'(z,t)]_k^j\,S_{\alpha_\xi}(z,t)\ \hat{\Sigma}_j((\alpha_\xi(z,t))_3)\ S_{\alpha_\xi}^{-1}(z,t), \tag{4.6}$$

with S_{α_ξ} as in Eqs. (2.14) and (3.16). Note that $\hat{\Sigma}_k$ in Eq. (4.4) depends on the t coordinate in H^3. The projection operators with respect to the triad vectors $\mathbf{e}_i(z,t;\xi)$ are simply

$$\Sigma^P(z,t,\xi,i):=\Sigma_k\ e_i^k=\frac{\hbar}{2}S_{\alpha_\xi}(z,t)\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}S_{\alpha_\xi}^{-1}(z,t). \tag{4.7}$$

Now, what is the meaning of Eq. (4.7)? We consider an elementary wave excited at $(\xi,0)$, i.e. a linear combination of two spinors in Eq. (3.17) with $k=\pm 1$, and all other variables identical. The wave fronts are the horospheres $P(z,t;\xi)=\text{const}$, the unit wave vectors on the wave fronts are just $\mathbf{e}_3(z,t;\xi)$ or $-\mathbf{e}_3(z,t;\xi)$. The expectation values of Σ^P with respect to this elementary wave define the three components of its spin vector in the oriented basis frames $\mathbf{e}_i(z,t;\xi)$ on the wave front:

We define with the ψ in Eq. (3.17) a mixed spin state

$$\psi(a,b,s,\xi,i):=a\psi(s,\xi,z,t,\tau,i,1)+b\psi(s,\xi,z,t,\tau,i,-1) \tag{4.8}$$

and calculate by means of Eqs. (2.7) and (3.21) the expectation value of $\Sigma^P(z,t,\xi,k)$

$$\langle\Sigma^P(k)\rangle_\psi:=\frac{\langle\psi(a,b,s,\xi,i),\Sigma^P(z,t,\xi,k)\psi(a,b,s,\xi,i)\rangle}{\langle\psi,\psi\rangle}. \tag{4.9}$$

Though the ψ are not square integrable, it is easy to see that $\langle\psi,\Sigma^P\psi\rangle$ is "proportional" to $\langle\psi,\psi\rangle$. This can be easily made rigorous by using some limit procedure with respect to the volume integration in the scalar product.

We obtain

$$\langle\Sigma^P(1)\rangle_\psi=\hbar\frac{\mathrm{Re}(\bar{a}b)h(s,\tau)}{|a|^2+|b|^2},\quad \langle\Sigma^P(2)\rangle_\psi=\hbar\frac{\mathrm{Im}(\bar{a}b)h(s,\tau)}{|a|^2+|b|^2},\quad \langle\Sigma^P(3)\rangle_\psi=\frac{\hbar}{2}\frac{|a|^2-|b|^2}{|a|^2+|b|^2}, \tag{4.10}$$

with the transversal polarization factor

$$h(s,\tau):=\frac{s^2c^2R^{-2}\varphi(s^2,\tau)\overline{\varphi(s^2,\tau)}-D_\tau^-\varphi(s^2,\tau)\overline{D_\tau^-\varphi(s^2,\tau)}}{s^2c^2R^{-2}\varphi(s^2,\tau)\overline{\varphi(s^2,\tau)}+D_\tau^-\varphi(s^2,\tau)\overline{D_\tau^-\varphi(s^2,\tau)}}. \tag{4.11}$$

Remarks: (1) The averages do not depend on the spectral parameters ξ and i in the wave functions. (2) The transversal polarizations in the $\mathbf{e}_1(z,t;\xi)$ and $\mathbf{e}_2(z,t;\xi)$ directions vanish if a or b is zero. (3) The longitudinal polarization $\langle\Sigma^P(3)\rangle_\psi$ is conserved. (4) The ψ in Eq. (3.17) constitute a complete set of eigenfunctions, which are longitudinally polarized. (5) If we choose for (a,b) in Eq. (4.8) the four pairs $(1,1)/\sqrt{2}$, $(1,-1)/\sqrt{2}$, $(1,i)/\sqrt{2}$, $(1,-i)/\sqrt{2}$, we get a complete set of transversally polarized elementary waves. (6) This description of spin carries easily over to multiply connected spaces, cf. Secs. VII and VIII. All we have to do is to project the horospheres[18] together with the spin operators and the vector fields defined on them into the fundamental polyhedron representing the three-manifold. We will not elaborate on this here.

V. PARITY AND OTHER DISCRETE SYMMETRIES IN HYPERBOLIC SPACE H^3

A space inversion P in H^3 may be realized as

$$P(z,t):=\frac{1}{|z|^2+t^2}(-z,t). \tag{5.1}$$

The center of this reflection is (0,1). [In this section we put the curvature radius $R=1$, cf. Eq. (A1).] The geometric meaning of $P(z,t)$ is that the point (0,1) lies always in the middle of the geodesic arc joining (z,t) and $P(z,t)$. In the B^3 model of hyperbolic geometry a space reflection is $P(x)=-x$, and $P(z,t)$ is obtained from this by mapping B^3 onto H^3 via Eq. (A10) of Ref. 17. [We denote the space reflection with the same symbol as the Poisson kernel in Eq. (3.19), hopefully no confusion will arise.]

We have

$$P=R\circ I=I\circ R=R(\bar{z},t), \tag{5.2}$$

where $R(z,t)$ is the Möbius transformation generated, cf. Eq. (2.15), by the matrix R in Eq. (2.16), and $I(z,t):=(\bar{z},t)$.

The reflection $I(z,t)$ leaves the metric (A2) invariant, moreover, a matrix S_I is readily found so that $S_I\psi(\bar{z},t)$ is a solution of Eq. (2.1) if $\psi(z,t)$ is. Clearly, we see from Eq. (2.2) that S_I must commute with $\tilde{\gamma}_0,\tilde{\gamma}_1,\tilde{\gamma}_3$, and anticommute with $\tilde{\gamma}_2$, and accordingly we may choose S_I proportional to $\tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_3$. We put

$$S_I:=-i\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}. \tag{5.3}$$

Applying Eq. (2.17), we obtain the transformation matrix S_P of the space inversion P, $S_P(z,t):=S_R(z,t)S_I=S_IS_R(\bar{z},t)$, where S_R is given by Eqs. (2.18) and (2.14). Thus we have

$$S_P(z,t)=\begin{pmatrix}\hat{S}_P & 0\\ 0 & -\hat{S}_P\end{pmatrix},\quad \hat{S}_P(z,t):=\frac{1}{\sqrt{|z|^2+t^2}}\begin{pmatrix}t & -\bar{z}\\ z & t\end{pmatrix}. \tag{5.4}$$

Clearly S_P satisfies Eqs. (2.12) and (2.13); $S_P\psi(P(z,t))$ is a solution of Eq. (2.1) if $\psi(z,t)$ is.

Remark: H^3 is a homogeneous space, and the point (0,1) is in no way distinguished. We have chosen it as the center of reflection just for technical convenience. A space inversion around any other point (z_0,t_0) in H^3 may be realized by $(z,t)\to P_{(z_0,t_0)}(z,t):=M^{-1}_{(z_0,t_0)}\circ P\circ M_{(z_0,t_0)}(z,t)$, where $M_{(z_0,t_0)}$ is any Möbius transformation (2.15) that maps (z_0,t_0) into (0,1). From the geometric meaning of $P(z,t)$ it is clear that (z_0,t_0) lies always in the middle of the geodesic arc joining (z,t) and $P_{(z_0,t_0)}(z,t)$, and thus the isometry $(z,t)\to P_{(z_0,t_0)}(z,t)$ is evidently independent of the choice of $M_{(z_0,t_0)}(z,t)$. Combining Eqs. (2.10) and (5.4) we obtain the spinor representation $P_{(z_0,t_0)}$ of the space inversion as

$$\psi(z,t)\to P_{(z_0,t_0)}\psi(z,t):=S_{M_{(z_0,t_0)}}(z,t)\,S_P(M_{(z_0,t_0)}(z,t))\,S_{M^{-1}_{(z_0,t_0)}}(P\circ M_{(z_0,t_0)}(z,t))\,\psi(P_{(z_0,t_0)}(z,t)).$$

Using Eqs. (5.2), (5.3), and (2.17) we have $P^2_{(z_0,t_0)}=\pm 1$, the sign depending on the choice of the SL(2,C) matrices representing $M^{\pm 1}_{(z_0,t_0)}$ in $S_{M^{\pm 1}_{(z_0,t_0)}}$, cf. the end of Sec. II. Anyway, we may still include an arbitrary phase factor in the definition of $P_{(z_0,t_0)}$. In the following we will assume, without loss of generality, that the center of reflection is (0,1), and that the space reflection is realized by P: $\psi\to S_P\psi(P(z,t))$.

The operators for charge conjugation and time reversal are readily defined, quite similarly as in Minkowski space.[14] For the charge conjugation we choose

$$C:\quad \psi\to S_C\psi^*,\quad S_C:=-\tilde{\gamma}_2. \tag{5.5}$$

(We denote the complex conjugate of spinors and matrices by an asterisk.) One checks readily that this is a symmetry of Eq. (2.2).

A little more delicate is the definition of the time reversal. If the expansion factor in Eq. (A1) is generically time dependent, there cannot be a time symmetry in any kind of physical evolution. Nevertheless there is quite a simple, though formal way to define a time inversion, $T: \tau \to -\tau$, if we extend the time axis $[0, \infty]$ on which $a(\tau)$ is defined to $[-\infty, \infty]$, and put $a(\tau)=a(-\tau)$. [There is no reason at this point to impose smoothness conditions on $a(\tau)$ at $\tau=0$]. This means that we consider a twin universe on $[-\infty, 0]$ that is contracting, the contraction being time symmetric to the expansion in $[0, \infty]$.

So, for example, $\psi \to \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\psi(-\tau)$ is a geometric symmetry of Eq. (2.2), and the Wigner time reversal is

$$T:\psi \to S_T\psi^*(-\tau), \quad S_T:=\tilde{\gamma}_1\tilde{\gamma}_3. \tag{5.6}$$

Next we calculate the action of P, C, T on the basis functions (3.17). With Eqs. (5.3), (5.4), and (2.17) we have, suppressing variables that are not acted upon

$$P\psi(s,\xi,z,t)=S_P(z,t)S_{\alpha_\xi}(P(z,t))\hat{\psi}((\alpha_\xi\circ P(z,t))_3)=S_I S_{\alpha_\xi\circ R}(\bar{z},t)\hat{\psi}((\alpha_\xi\circ R(\bar{z},t))_3), \tag{5.7}$$

with $S_{\alpha_\xi\circ R}$ as defined in Eqs. (2.14) and (2.18). Furthermore, $S_I S_{\alpha_\xi\circ R}(\bar{z},t) = -S_{\alpha_{\bar{\xi}}\circ R}(z,t)S_I$.

For C we obtain

$$C\psi(s,\xi,z,t)=S_C S^*_{\alpha_\xi}(z,t)\hat{\psi}^*((\alpha_\xi(z,t))_3)=S_{\alpha_\xi}(z,t)S_C\,\hat{\psi}^*((\alpha_\xi(z,t))_3) \tag{5.8}$$

and similarly for T. In fact, for an arbitrary Möbius transformation h, we always have $S_{C,T}S_h^*=S_h S_{C,T}$.

With Eqs. (5.7) and (5.8) and the $\hat{\psi}$ in Eqs. (3.11)–(3.14) we calculate easily, suppressing the arguments z,t, τ

$$P\psi(s,\xi,i,k)=(-1)^{(i+k)/2+1}\xi^{(1-k)/2}\bar{\xi}^{(1+k)/2}|\xi|^{-2(3/2+is)}\psi(s,-\bar{\xi}^{-1},i,-k), \tag{5.9}$$

$$C\psi(s,\xi,i,k)=(-1)^{(i+k)/2}\psi(-s,\xi,-i,-k), \tag{5.10}$$

$$T\psi(s,\xi,i,k)=k\psi(-s,\xi,i,-k). \tag{5.11}$$

The phase factors in P,C,T we have chosen so that $P^2=1$, $C^2=1$, $T^2=-1$. C and T commute, T and P commute, C and P anticommute. C, P, T are (anti-)unitary with respect to the scalar product (2.7). [The ξ factors in Eq. (5.9) are needed to restore the arguments in the δ function in Eq. (3.21).]

Remarks:

(1) If we replace the $\psi(a,b,s,\xi,i)$ in Eqs. (4.9) and (4.10) by $P\psi(a,b,s,\xi,i)$, then we have to make the changes $a \to -b\xi|\xi|^{-1}$, $b \to a\bar{\xi}|\xi|^{-1}$, on the right hand side of the formulas in Eq. (4.10). Therefore we have for the expectation value of the longitudinal polarization $\langle\Sigma^P(3)\rangle_{P\psi}=-\langle\Sigma^P(3)\rangle_\psi$, whereas the transversal component of the spin vector is just rotated in the tangent planes [generated by $e_1(z,t;\xi)$, $e_2(z,t;\xi)$] by an angle depending on ξ. Also note that the wave front $P(z,t;\xi)=c$ is reflected into $P(z,t;-\bar{\xi}^{-1})=|\xi|^2c$. The triad (4.3) reverses its orientation, if it is P reflected, but so that the reflected e_3 still points to the outwards of the horospheres. The statement that P reverses the wave vector, cf. after Eq. (4.3), but leaves the orientation of the spin vector (4.10) unchanged, which is valid in Minkowski space, looses its meaning here: P reflects a hedgehog into a hedgehog (of a different size), some of their spines are oriented against each other (along geodesics), but others are not. Apart from this P, C, T are very much designed after their counterparts in Minkowski space.

(2) If $\mu=0$ in Eq. (2.1), then Eq. (3.10) can be integrated without explicit knowledge of $a(\tau)$; a positive frequency solution is

$$\varphi(s^2,\tau)=a^{-3/2}(\tau)\exp\left[-i|s|cR^{-1}\int^{\tau} a^{-1}(\tau)d\tau\right]. \tag{5.12}$$

With

$$\gamma_5:=\frac{i}{4!}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta=i\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3=\begin{pmatrix}0 & id\\ id & 0\end{pmatrix} \tag{5.13}$$

and with the ψ in Eq. (3.17) we have, suppressing variables

$$\gamma_5\psi(s,i,k)=(-1)^{(i+k)/2+1}\,\mathrm{sign}(s)\psi(s,i,k). \tag{5.14}$$

It is evident that the subspace of left-handed neutrinos and right-handed antineutrinos, generated by $\{\psi(|s|,1,-1),\ \psi(-|s|,-1,-1),\ \psi(-|s|,1,1),\ \psi(|s|,-1,1),\ s$ ranging over $R\}$, is invariant with respect to the transformations (2.10). Also T in Eq. (5.11) and the combined operation *CP*, cf. Eqs. (5.9) and (5.10), leave this subspace invariant.[14]

VI. ENERGY

The functional (2.8) determines the time evolution of the energy for the elementary wave fields ψ in Eq. (3.17). [At the end of this section we will discuss why we can call $E(\psi)$ in Eq. (2.8) energy.] The spectral parameter s in Eqs. (3.11)–(3.14) is real, cf. Eq. (3.24). Though these fields are not square integrable it is easy to see that the denominator in Eq. (2.8) is formally proportional to $\langle\psi,\psi\rangle$, and therefore this factor drops out, quite similarly as in Eq. (4.9).

First of all

$$E(\psi(s,\tau,1,1))=E(\psi(s,\tau,1,-1))=-E(\psi(s,\tau,-1,1))=-E(\psi(s,\tau,-1,-1)). \tag{6.1}$$

E depends only on τ, the spectral parameter s, and the energy index. Therefore it is enough to consider the $\hat{\psi}$ in Eq. (3.11).

From Eqs. (2.9), (3.11), and (3.17) we calculate easily

$$E(\psi(s,\tau,1,1)):=E(\varphi(s^2,\tau)):=\frac{\hbar i}{2}\frac{U(\varphi)}{W(\varphi)}, \tag{6.2}$$

with φ satisfying Eq. (3.10), W as in Eq. (3.8), and

$$U(\varphi):=2\,\mathrm{Im}\left\{\overline{D_\tau^-\varphi(s^2,\tau)}\,\frac{d}{d\tau}(D_\tau^-\varphi(s^2,\tau))+s^2c^2R^{-2}\overline{\varphi(s^2,\tau)}\dot{\varphi}(s^2,\tau)\right\}. \tag{6.3}$$

Remarks:

(1) A second independent solution of Eq. (3.10) is $D_\tau^+\bar{\varphi}$, which gives just $-E$.

(2) The energy of (anti-)particles is perfectly well defined by Eq. (6.2), provided we can disentangle positive and negative frequencies. In a rapidly varying background metric that is usually not the case.[21,22] In a period of rapid variation of the expansion factor the solutions of the wave equation may not exhibit any kind of periodicity, cf. Example 5 below. In this case the particle/antiparticle concept breaks down, and there is no point to attach an energy to wave solutions in such regimes.

(3) In the Examples 3–5 below it is quite straightforward to identify positive/negative frequencies, just by analogy to Minkowski space. We can choose asymptotic fundamental solutions of Eq. (3.10) which contain a factor $\exp(\pm if(\tau))$, $f(\tau)$ a strictly increasing function, $f(\tau)\to\infty$, in the respective limits $\tau\to\infty,0$. That we have made the right choice will finally become clear when we compare the energies obtained in Examples 1–5 with the corresponding energies of classical geodesic particles, cf. the end of this section.

For practical purposes it is convenient to express U, W in Eq. (6.2) as functionals of $\hat{\varphi}:=a^2(\tau)\varphi$. Instead of Eq. (3.10) we have

$$\ddot{\hat{\varphi}}+\left[-\frac{1}{2}\frac{\ddot{a}}{a}+i\frac{\dot{a}}{a}\mu c+\frac{1}{4}\frac{\dot{a}^2}{a^2}+\mu^2c^2+\frac{s^2c^2}{a^2R^2}\right]\hat{\varphi}=0, \tag{6.4}$$

the W in Eq. (3.8) reads

$$W(\varphi)=a^{-2}(\tau)\left\{\dot{\hat{\varphi}}\,\dot{\bar{\hat{\varphi}}}-\frac{1}{2}\frac{\dot{a}}{a}(\bar{\hat{\varphi}}\,\dot{\hat{\varphi}}+\hat{\varphi}\dot{\bar{\hat{\varphi}}})-i\mu c(\bar{\hat{\varphi}}\,\dot{\hat{\varphi}}-\hat{\varphi}\dot{\bar{\hat{\varphi}}})+\hat{\varphi}\bar{\hat{\varphi}}\left[\frac{1}{4}\frac{\dot{a}^2}{a^2}+\mu^2c^2+\frac{s^2c^2}{a^2R^2}\right]\right\} \tag{6.5}$$

and the U in Eq. (6.3) is

$$U(\varphi)=2i\mu cW(\varphi)+\frac{2s^2c^2}{a^4(\tau)R^2}(\bar{\hat{\varphi}}\,\dot{\hat{\varphi}}-\hat{\varphi}\dot{\bar{\hat{\varphi}}}). \tag{6.6}$$

We denote by $\hat{\varphi}_\pm$ positive/negative frequency solutions, whenever that is possible.

Example 1: $a(\tau)=1$. We have

$$\hat{\varphi}_\pm=\exp(-i\omega_\pm\tau),\quad \omega_\pm:=\pm\sqrt{\mu^2c^2+s^2c^2R^{-2}},\quad E=\hbar\omega_+. \tag{6.7}$$

Example 2: Neutrinos, cf. Eqs. (5.12)–(5.14). $a(\tau)$ is arbitrary. We have

$$\hat{\varphi}_\pm=a^{1/2}(\tau)\exp\left[-i\omega_\pm\int^\tau a^{-1}(\tau)d\tau\right],\quad \omega_\pm:=\pm|s|cR^{-1},\quad E=\frac{\hbar\omega_+}{a(\tau)}. \tag{6.8}$$

Example 3: $a(\tau)=\Lambda\tau$, $\Lambda:=c/R$, R is the curvature radius as in Eq. (A2). We put $\omega:=\mu c$. A solution of Eq. (6.4) is the Whittaker function $\hat{\varphi}=W_{1/2,is}(2i\omega\tau)$, a second fundamental solution is obtained by replacing 1/2 by $-1/2$, and τ by $-\tau$. In the late stage of the evolution, $\tau\to\infty$, we have

$$\hat{\varphi}=\exp[-i\omega\tau](2i\omega\tau)^{1/2}\left[1+\frac{is^2}{2\omega}\frac{1}{\tau}-\frac{s^2(1+s^2)}{8\omega^2}\frac{1}{\tau^2}+O(\tau^{-3})\right]. \tag{6.9}$$

Clearly $\hat{\varphi}$ is a positive frequency solution in this regime, and

$$E=\hbar\omega+\frac{\hbar\Lambda^2s^2}{2\omega}\frac{1}{a^2(\tau)}+O(a^{-4}(\tau)). \tag{6.10}$$

Evidently Eqs. (6.9) and (6.10) hold true for all expansion factors $a(\tau)\sim\Lambda\tau$.

Next we look at the behavior of $\hat{\varphi}$ in the early stage of the expansion, $\tau\to 0$

$$\hat{\varphi}=A(s)\tau^{1/2-is}+A(-s)\tau^{1/2+is}+O(\tau^{3/2}),\quad A(s):=\frac{\Gamma(2is)}{\Gamma(is)}(2i\omega)^{1/2-is}. \tag{6.11}$$

The positive frequency solution (6.9) has gotten an admixture of negative frequencies during the cross-over to small τ values. This is a nice example for particle creation due to the time variation of the expansion factor.[2] For an expansion factor $a(\tau)\sim\Lambda\tau$ we have

$$\hat{\varphi}_{\pm}=(\Lambda\tau)^{1/2\mp i|s|}+O(\tau^{3/2}),\quad E\sim\hbar|s|\tau^{-1}. \tag{6.12}$$

Example 4: $a(\tau)\sim(\Lambda\tau)^{\alpha}$, $\tau\to\infty$, $0<\alpha<\infty$. With $\omega:=\mu c$ we have for $\alpha>1/2$

$$\hat{\varphi}_{+}=(\Lambda\tau)^{\alpha/2}\exp(-i\omega\tau)\left\{1-\frac{i\Lambda s^2(\Lambda\tau)^{1-2\alpha}}{2\omega(1-2\alpha)}-\frac{\Lambda^2s^2(\Lambda\tau)^{-2\alpha}}{8\omega^2}-\frac{\Lambda^2s^4(\Lambda\tau)^{2-4\alpha}}{8\omega^2(1-2\alpha)^2}\right.$$
$$\left.+O(\tau^{-1-2\alpha},\tau^{3-6\alpha})\right\},$$

for $\alpha=1/2$ we have $\hat{\varphi}_{+}\sim W_{(1/4-i\Lambda s^2/(2\omega)),1/4}(2i\omega\tau)$, and finally for $\alpha<1/2$

$$\hat{\varphi}_{+}=(\Lambda\tau)^{\alpha/2}\exp\left[-i\omega\tau-\frac{i\Lambda s^2(\Lambda\tau)^{1-2\alpha}}{2\omega(1-2\alpha)}+O(\tau^{1-4\alpha})\right]\left[1-\frac{\Lambda^2s^2(\Lambda\tau)^{-2\alpha}}{8\omega^2}+O(\tau^{-1-2\alpha})\right].$$

In all these cases we obtain again formula (6.10) [with a $(\tau)\sim(\Lambda\tau)^{\alpha}$] for the energy. Einstein's equations suggest that $\alpha=1$ in this limit, compare however the comments in the introduction of Ref. 8.

Example 5: Finally we discuss the approach to the initial singularity, the time asymptotics being a $(\tau)\sim(\Lambda\tau)^{\alpha}$, $\tau\to0$, $0<\alpha<\infty$. The case $\alpha=1$ has been treated in Example 3. If $0<\alpha<1$, then we have $\hat{\varphi}\sim(\Lambda\tau)^{\alpha/2}$ and $\hat{\varphi}\sim(\Lambda\tau)^{1-\alpha/2}$ as fundamental solutions of Eq. (6.4). Clearly these solutions are not periodic, and positive/negative frequencies cannot be defined, and neither can particles/antiparticles; compare the Remarks after Eq. (6.3).

If however $\alpha>1$, we have

$$\hat{\varphi}_{\pm}\sim(\Lambda\tau)^{\alpha/2}\exp\left[\mp\frac{i|s|}{\alpha-1}(\Lambda\tau)^{1-\alpha}\right],\quad E\sim\hbar\Lambda|s|(\Lambda\tau)^{-\alpha}. \tag{6.13}$$

Let us discuss the foregoing examples a little. We want to justify why the quantity $E(\psi)$ defined in Eq. (2.8) is the energy of the wave field. (I would like to thank the referee for raising this important question). Note at first that the identity $T^{\mu\nu}{}_{;\nu}=0$ [which of course holds true for $T_{\mu\nu}$ in Eq. (2.9) with solutions ψ of Eq. (2.1)], does not supply in general Riemannian space–time conserved quantities, cf. Ref. 26. Though it is clear from the foregoing examples that $E(\psi)$ is not conserved, there is nevertheless compelling evidence to call it energy in the context of the RW models considered here.

In the case of conformally coupled fields we have the conservation law $a(\tau)E(\psi)=\text{const}$, cf. Example 2, or in the electromagnetic case Eq. (4.6) in Ref. 8. That is of course common knowledge and related to the existence of a conformal, timelike Killing vector field on the manifold, cf. Refs. 21 and 22.

To convince ourselves that $E(\psi)$ is also the correct energy formula for massive particles, we compare Eq. (2.8) in adiabatic regimes of the expansion [we have $(d^n/d\tau^n)(\dot{a}(\tau)/a(\tau))\to0$, for $\tau\to\infty$, in Example 4], with the energy formula for classical, geodesically moving particles, cf. Eq. (2.9) in Ref. 17.

Before we do that note that the space part of the horospherical action, cf. Eqs. (2.20) and (2.21) of Ref. 17, appears in the phase of the horospherical elementary waves, cf. Eq. (3.18). We only have to identify the spectral parameter s in Eq. (3.18) with the integration constant

ν in S_0^b, according to formula (4.15) of Ref. 17. In fact, we may write then $P^{is}=\exp(-(i/\hbar)S_b^0)$. (Analogous equations hold also true in other homogeneous spaces, cf. Ref. 27; in this sense the semiclassical approximation for geodesic motion is exact.)

In the same way we express ν by s in the energy formula for classical particles, Eq. (2.9) of Ref. 17. Expanding the root there in powers of $a^{-2}(\tau)$, we arrive at Eq. (6.10). We conclude that if we define the energy of the massive wave field by Eq. (2.8) it coincides [to the asymptotic order given in Eq. (6.10)] with the energy of geodesically moving particles in the final stage of the expansion, cf. Examples 1 and 4.

Let us finally look at Example 5, Eq. (6.13), and the second part of Example 3, Eq. (6.12). Though the variation of $a(\tau)$ cannot be regarded as slow, we get asymptotic identity between the energy of the field and the energy of the classical particle by the identification of ν and s. In the leading asymptotic order the mass does not enter, and therefore we get here the same result for the energy as in Example 2.

Remark: This section carries over as it stands to the case of spinor fields in multiply connected RW cosmologies as discussed in the next two sections. The reason for this is that—contrary to scalar wave equations[7]—the spectrum of the Dirac equation remains unaltered, there are no bound states emerging out of the infinite space, cf. Sec. VII. In Eqs. (2.7) and (2.8) we have then to replace the integration over H^3 by an integration over the fundamental polyhedron F of the three-manifold, cf. Sec. IV of Ref. 7. Otherwise the same reasoning applies as at the beginning of this section, if we replace ψ by ψ^Γ defined later in Eq. (7.12). Thus the energy of the wave fields ψ^Γ on the multiply connected three-manifold is given by the formulas in this section. The energy formula for classical particles remains also unaltered, because the geodesics inherit of course the time parametrization and the energy from their covering trajectories.

VII. DIRAC'S EQUATION IN OPEN, MULTIPLY CONNECTED RW COSMOLOGIES

Open RW cosmologies with multiply connected spacelike slices of constant negative curvature have been extensively discussed in Refs. 7–9 and 18, to which we refer for the basic concepts. In particular, the three-space is represented as a fundamental polyhedron F in H^3, whose covering group Γ is a discrete subgroup of the group of Möbius transformations defined in Eq. (2.15). To every such transformation $h(z,t)$ there correspond exactly two SL(2,C) matrices $\pm\hat{h}$. By $\hat{\Gamma}$ we denote the discrete subgroup of SL(2,C) consisting of all matrices $\hat{h}$, with $h(z,t)\in\Gamma$. As pointed out at the end of Sec. II, Eq. (2.10) provides a representation for $\hat{\Gamma}$, but only a projective one for Γ.

In order to obtain an unambiguous representation of Γ we introduce a character system[28–30] on $\hat{\Gamma}$. To every $\hat{g}\in\hat{\Gamma}$ we assign a complex number $\chi(\hat{g})$, $|\chi(\hat{g})|=1$, so that for all $\hat{g},\hat{h}\in\hat{\Gamma}$

$$\chi(\hat{g}\circ\hat{h})=\chi(\hat{g})\chi(\hat{h}),\quad \chi(-\hat{g})=-\chi(\hat{g}),\quad \chi(\hat{\mathrm{id}})=1. \tag{7.1}$$

At the end of this section we sketch an algorithmic construction of such character systems for covering groups of three-manifolds that are topologically either solid handlebodies $I\times D_N$, or thickened Riemann surfaces $I\times S_g$. Here I is a finite open interval, D_N a disk with $N\geqslant 1$ smaller disks removed, and S_g is a Riemann surface of genus $g\geqslant 2$. In the first case the covering group Γ is a Schottky group, in the second case it is quasi-Fuchsian. In fact, the characters can be chosen real, taking only the values ± 1. That we will assume in the following, namely, $\chi=\bar{\chi}$.

Remark: Most of the results obtained in Refs. 7,8,18 and this article hold presumably also true for other hyperbolic manifolds that admit algebraically and geometrically finite Kleinian groups of the second kind without parabolic elements as covering groups. Parabolic elements give rise to cusp singularities on the manifolds that can alter the spectrum of the wave equations dramatically, for example, bound states embedded in the continuous spectrum may occur, cf., e.g., Ref. 30. "Of the first kind" means that the limit set $\Lambda(\Gamma)$ fills the whole compactified complex plane, "of the second kind" means that this is not the case. Groups of the first kind correspond to manifolds of finite volume. The spectrum of the wave equations on them is of course discrete, provided that there are no parabolic cusps. The condition "algebraically finite" means that Γ is finitely generated, "geometrically finite" means that there is a fundamental polyhedron in H^3 with a finite number of faces, the first does not always imply the second. These finiteness conditions can presumably be relaxed, but there are many technical problems to solve, and the three-manifolds as well as their covering groups can then get very bizarre, cf. Refs. 10 and 25. I have formulated and checked the results of this article and the preceding ones for hyperbolic three-manifolds with quasi-Fuchsian and Schottky covering groups, which do not contain elliptic and parabolic elements, and which are algebraically and geometrically finite. These manifolds constitute two generic classes of hyperbolic three-manifolds, whose topology can be easily visualized.

We define, cf. Eq. (2.14), for $h\in\Gamma$

$$S_h^\chi:=\chi(\hat{h})S_{\hat{h}}. \tag{7.2}$$

Clearly S_h^χ is unambiguous, independent of the sign of $\hat{h}$. Furthermore Eq. (2.17) holds true with S_h replaced by S_h^χ. Accordingly, cf. Eq. (2.10)

$$\psi\rightarrow S_h^\chi(z,t)\psi(h(z,t)) \tag{7.3}$$

defines a representation of Γ.

We construct now a complete orthogonal set of eigenfunctions of Eq. (2.1) on the three-space manifold (F,Γ). That can be done by projecting the H^3-eigenfunctions (3.17) into the polyhedron F, which amounts to periodize them with respect to the covering group Γ, using the representation (7.3). Quite similar procedures we have used in Ref. 8, and so we sketch that very briefly, pointing out just some technical details connected with spinors.

With the ψ in Eq. (3.17) we construct formally the Eisenstein series[29,30]

$$\psi^\Gamma(s,\xi,z,t,i,k):=\sum_{\gamma\in\Gamma}S_\gamma^\chi(z,t)\psi(s,\xi,\gamma(z,t),i,k). \tag{7.4}$$

We have for all $g\in\Gamma$, suppressing spectral parameters

$$S_g^\chi(z,t)\psi^\Gamma(g(z,t))=\psi^\Gamma(z,t). \tag{7.5}$$

Therefore ψ^Γ is a generalized eigenfunction on the manifold (F,Γ), provided that the series (7.4) converges nicely.

To derive the orthogonality and completeness relations on F we let the γ in Eq. (7.4) act on ξ in the complex plane, rather than on (z,t) in H^3. To this end we write the series (7.4) more explicitly, using Eqs. (3.11)–(3.14) and (2.17)

$$\psi^\Gamma=\sum_{\gamma\in\Gamma}\chi(\gamma)S_{\alpha_\xi\circ\gamma}(z,t)\hat{\psi}(s,(\alpha_\xi\circ\gamma(z,t))_3,i,k). \tag{7.6}$$

For Möbius transformations the symbol $g\circ h$ means just $g(h(z,t))$.

Now, in terms of SL(2,C) matrices, we have

$$\alpha_\xi\circ\gamma=L_\gamma\circ\alpha_{\gamma^{-1}\xi},\quad \alpha_\xi:=\begin{pmatrix}0 & i\\ i & -i\xi\end{pmatrix},\quad \gamma:=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad L_\gamma:=\begin{pmatrix}(a-c\xi)^{-1} & c\\ 0 & (a-c\xi)\end{pmatrix}. \tag{7.7}$$

Therefore we may use in Eq. (7.6)

$$S_{\alpha_\xi \circ \gamma}(z,t)=S_{\alpha_{\gamma^{-1}\xi}}(z,t)S_{L_\gamma}, \tag{7.8}$$

which follows from Eq. (2.17); the S are defined by Eqs. (2.14) and (2.18). The action of S_{L_γ} on the $\hat{\psi}$ in Eqs. (3.11)–(3.14) we can express in terms of automorphic factors[28]

$$j_\gamma(\xi):=\frac{c\xi+d}{|c\xi+d|}, \tag{7.9}$$

with γ as in Eq. (7.7). Clearly, $|j_\gamma(\xi)|=1$, $j_{h\circ g}(\xi)=j_g(\xi)j_h(g(\xi))$, with g, h acting as Möbius transformations in the complex plane [$t=0$ in Eq. (2.15)]. Applying S_{L_γ} onto $\hat{\psi}$ we obtain

$$S_{\alpha_\xi \circ \gamma}(z,t)\hat{\psi}(i,k)=j_\gamma^k(\gamma^{-1}\xi)S_{\alpha_{\gamma^{-1}\xi}}(z,t)\hat{\psi}(i,k). \tag{7.10}$$

The t component of $\alpha_\xi(z,t)$ is just the Poisson kernel, $(\alpha_\xi(z,t))_3=P(z,t;\xi)/R$, cf. Eq. (3.19). We have, cf. Eq. (A23) in Ref. 17

$$P(\gamma(z,t);\xi)=P(z,t;\gamma^{-1}\xi)\,|\gamma^{-1\prime}\xi|, \tag{7.11}$$

with γ as in Eq. (7.7), and $\gamma^{-1\prime}(\xi)=(c\xi-a)^{-2}$.

With Eqs. (7.10) and (7.11) we may write ψ^Γ as

$$\psi^\Gamma(s,\xi,z,t,i,k)=\sum_{\gamma\in\Gamma}\chi(\gamma)j_\gamma^k(\gamma^{-1}\xi)\,|\gamma^{-1\prime}\xi|^{1+is}\psi(s,\gamma^{-1}\xi,z,t,i,k); \tag{7.12}$$

the series (7.6) and (7.12) are term by term identical.

Let us denote by $\cup_i f_i$ the collection of free faces of F, cf., e.g., Refs. 31 and 24. They constitute a fundamental domain of Γ in the complex plane. The series (7.12) provide a complete orthogonal system for the Dirac equation on (F,Γ), with ξ ranging in $\cup_i f_i$, s ranging in **R**, and $i,k=\pm1$, see below. The scalar product is given by Eq. (2.7) with the domain of integration H^3 replaced by F.

Let us regard for the moment ψ^Γ in Eq. (7.12) as a function of the complex variable s, and assume $\xi\in\cup_i f_i$. Because ξ is outside the limit set $\Lambda(\Gamma)$ (Refs. 25 and 31), we can majorize every component ψ_α^Γ of the spinors (7.12) by

$$|\psi_\alpha^\Gamma|<\text{const}\sum_{\gamma\in\Gamma}|\gamma^{-1\prime}\xi|^{1+is} \tag{7.13}$$

quite similarly as we did with the electromagnetic potentials in Ref. 8. For $\mathrm{Re}(1+is)>\delta$, δ the Hausdorff dimension of $\Lambda(\Gamma)$, this series converges.[32] If $\delta>1$ we define ψ^Γ for real s by analytic continuation. Contrary to the scalar case,[7] where the wave field is essentially a Dirichlet series with positive terms, there is no pole at $s=i(1-\delta)$ in Eq. (7.12). [The series (7.13) does have a pole there. Compare also Ref. 33 where the analyticity properties of series of a similar type are discussed.] In fact, such a pole would correspond to a bound state wave field of the Dirac equation with a purely imaginary s, and that would violate the conservation of probability: if s is imaginary, then $\psi^{\Gamma\dagger}\psi^\Gamma$ is no longer proportional to W in Eq. (3.8), and we cannot apply Eq. (3.9) after the volume integration.

The orthogonality and completeness relations on the manifold (F,Γ) are just Eqs. (3.21) and (3.24), if we make the following replacements: ψ by ψ^Γ, the domain of integration H^3 in

Eq. (3.21) by F, and in Eq. (3.24) we integrate over $\mathbb{R}\times\cup_i f_i$. For details we refer to Ref. 8, where we used the same methods. But for reference in Sec. VIII we note

$$\int_F \psi^{\Gamma\dagger}\psi^{\Gamma}\, dV_{H^3}=\int_{H^3} \psi^{\Gamma\dagger}\psi\, dV_{H^3}$$

$$=\sum_{\gamma\in\Gamma}\chi(\gamma)j_{\gamma}^{-k}(\gamma^{-1}\xi)\,|\gamma^{-1'}\xi|^{1-is}\int_{H^3}\psi^{\dagger}(s,\gamma^{-1}\xi,z,t,i,k)\,\psi(s',\xi',z,t,m,n)\,dV_{H^3}$$

$$=\delta_{im}\delta_{kn}2\pi^3R^5a^3(\tau)\,W(s^2,\tau)\,\frac{1}{s^2}\,\delta(s-s')\sum_{\gamma\in\Gamma}\delta(\gamma^{-1}\xi-\xi')\chi(\gamma)$$

$$j_{\gamma}^{-k}(\gamma^{-1}\xi)\,|\gamma^{-1'}\xi|^{1-is}, \tag{7.14}$$

where we used Eqs. (7.5), (7.12), and (2.17), the invariance of the volume element under Γ, the tiling property $\cup_{\gamma\in\Gamma}\gamma(F)=H^3$, and finally Eq. (3.21). Because both ξ and ξ' range in the fundamental domain $\cup_i f_i$, the argument of the delta functions can only be zero if $\gamma=\mathrm{id}$, and therefore only this term in the Γ series counts.

A similar calculation for the completeness relation provides a Γ series with coefficients containing $\delta_{H^3}(z',t';\gamma^{-1}(z,t))$, cf. Eq. (A7) in Ref. 17, and here again, because (z',t') and (z,t) range in the fundamental domain F, only the term with $\gamma=\mathrm{id}$ counts.

Finally we construct the promised character system for Γ, cf. Eq. (7.1). Let F be a fundamental polyhedron with $2N$ faces identified by Möbius transformations T_n, $n=1,\ldots,N$. The T_n generate either a Schottky or quasi-Fuchsian covering group Γ, cf. the comments after Eq. (7.1) and Refs. 34 and 18. In the case of quasi-Fuchsian groups there exist relations[31] among the generators. In the example discussed in Secs. 5, 6 of Ref. 34 [see in particular Eqs. (5.4) and (5.6) there], we have $N=5$ and

$$T_5\circ T_2\circ T_1^{-1}\circ T_2^{-1}\circ T_1=\mathrm{id},\quad T_5^{-1}\circ T_4\circ T_3^{-1}\circ T_4^{-1}\circ T_3=\mathrm{id}, \tag{7.15}$$

with id as the identity Möbius transformation, and $T_i\circ T_k:=T_i(T_k(z,t))$.

Now, every transformations T_n is generated by an $SL(2,\mathbb{C})$ matrix, as explained at the beginning of this section. There are two such matrices, differing only by a sign. We choose arbitrarily one of them and denote it by $\hat{T}_n$. The relations (7.15) read now in terms of these matrices

$$\begin{aligned}&\hat{T}_5\circ\hat{T}_2\circ\hat{T}_1^{-1}\circ\hat{T}_2^{-1}\circ\hat{T}_1=+\hat{\mathrm{id}}\ \text{or}-\hat{\mathrm{id}},\\ &\hat{T}_5^{-1}\circ\hat{T}_4\circ\hat{T}_3^{-1}\circ\hat{T}_4^{-1}\circ\hat{T}_3=+\hat{\mathrm{id}}\ \text{or}-\hat{\mathrm{id}}.\end{aligned} \tag{7.16}$$

We denote by $\hat{\Gamma}$ the discrete matrix group generated by the $\hat{T}_n$, $n=1\cdots N$, and $-\hat{\mathrm{id}}$. To every generator $\hat{T}_n$ of $\hat{\Gamma}$ we attach a character $\chi(\hat{T}_n)$, taking the values $+1$ or -1, and require in addition

$$\chi(\hat{T}_n^{-1})=\chi(\hat{T}_n),\quad \chi(\hat{\mathrm{id}})=-\chi(-\hat{\mathrm{id}})=1. \tag{7.17}$$

Moreover we require that these characters satisfy relations corresponding to Eq. (7.16), namely,

$$\begin{aligned}&\chi(\hat{T}_5)\chi(\hat{T}_2)\chi(\hat{T}_1^{-1})\chi(\hat{T}_2^{-1})\chi(\hat{T}_1)=+1\ \text{or}-1,\\ &\chi(\hat{T}_5^{-1})\chi(\hat{T}_4)\chi(\hat{T}_3^{-1})\chi(\hat{T}_4^{-1})\chi(\hat{T}_3)=+1\ \text{or}-1.\end{aligned} \tag{7.18}$$

Every element $\hat{g}\in\hat{\Gamma}$ can be presented[31,35] as a product ("word") composed of the matrices ("letters") $\hat{T}_n^{\pm 1}$ and $\pm\hat{\mathrm{id}}$. We define the character $\chi(\hat{g})$ of $\hat{g}$ as the product of the characters of its letters. It is easy to see that this definition of $\chi(\hat{g})$ is unambiguous, it does not depend on the choice of the word for $\hat{g}$, because of Eqs. (7.16)–(7.18). Furthermore Eq. (7.1) holds true. In Ref. 34 we constructed an algorithm to calculate the elements of $\hat{\Gamma}$, generation by generation, as products of the letters $\hat{T}_n^{\pm 1}$. Simultaneously we calculate now their character.

Remark: The question is, if it is at all possible to choose a set of characters for the generating matrices with the required properties Eqs. (7.17), (7.18). In our example (7.15) this would be impossible if the $\hat{\mathrm{id}}$s in the two relations (7.16) have opposite signs. This would be unfortunate, because no spinor fields could then exist in our universes. In other words, if we eliminate the neck-transformation $\hat{T}_5$ in Eq. (7.16), then the resulting relation must have an $\hat{\mathrm{id}}$ with a plus sign. (Neck transformations can be eliminated, till finally only one relation remains, cf. the examples in Ref. 36.) This is indeed always the case, for Fuchsian groups a proof can be found in Ref. 37, see also Ref. 38. Quasi-Fuchsian groups we generate by continuous deformations of Fuchsian groups, which cannot change the sign of the relations. In Schottky groups there are no relations to satisfy.

VIII. INVERSIONS OF THE MULTIPLY CONNECTED THREE-SPACE AND *CP* VIOLATION

In Sec. V we constructed a reflection $P(z,t)$ of H^3, cf. Eq. (5.1), with fixed point $(z_0,t_0)=(0,1)$. In the three-space manifold (F,Γ) we can easily obtain in a very explicit way this space inversion by means of the universal covering projection $\pi^\Gamma(z,t)$, cf., e.g., Ref. 35. Note at first that because of the tiling property[31,24] of the fundamental polyhedron F, there is for almost every $(z,t)\in H^3$ a unique $\gamma\in\Gamma$, so that $(z,t)\in\gamma(F)$; "almost" refers here to points that lie on the polyhedral faces, the boundaries of the tiles. Therefore we can define

$$\pi^\Gamma:H^3\to F,\quad \pi^\Gamma(z,t)=\gamma^{-1}(z,t),\quad \text{if}\quad (z,t)\in\gamma(F). \tag{8.1}$$

On the boundary points of the tiles we define π^Γ by continuity; actually Eq. (8.1) is also unambiguous for boundary points.

The space reflection P^Γ we define as

$$P^\Gamma:F\to F,\quad P^\Gamma(z,t):=\pi^\Gamma(P(z,t)). \tag{8.2}$$

π^Γ projects the geodesic arc that connects (z,t) and $P(z,t)$ in H^3 into F. The fixed point of $P^\Gamma(z,t)$ is $\pi^\Gamma(0,1)$. For all $(z,t)\in F$ there is a geodesic joining (z,t) and $P^\Gamma(z,t)$, so that $d(P^\Gamma(z,t);\pi^\Gamma(0,1))=d(z,t;\pi^\Gamma(0,1))$, where $d(;)$ is the hyperbolic distance along this geodesic. Note that P^Γ is not bijective (though locally it is an isometry), because in a multiply connected space two points can be joined in several ways by geodesics, which are local minima of a global variational problem, for example, Fermat's variational principle.

Now, P^Γ maps geodesic arcs onto geodesic arcs [because $P(z,t)$ in H^3 is an isometry], and accordingly the geodesic equations for a classical particle moving in the three-space manifold are reflection invariant. For wave equations the perspectives are less favorable, because P^Γ is not measure (i.e., Riemannian volume) preserving. A finite strip, if reflected by P^Γ, can wrap around a handle of the manifold and overlap with itself. If this strip is the support of a wave packet, then the reflected wave interferes with itself,[18] and its norm will not be preserved.

Indeed, the Eisenstein series ψ^Γ in Eq. (7.4) is the canonical projection (by Γ periodization) of the H^3-basis function ψ into F. The reflection of ψ^Γ by P^Γ is just the canonical projection of $P\psi$, cf. Eq. (5.9), into F

$$P^\Gamma\psi^\Gamma(s,\xi,z,t,i,k)=\sum_{\gamma\in\Gamma} S_\gamma^\chi(z,t)S_P(\gamma(z,t))\psi(s,\xi,P\circ\gamma(z,t),i,k). \tag{8.3}$$

This can also be written, using Eqs. (5.9) and (7.12), as

$$P^\Gamma\psi^\Gamma(s,\xi,z,t,i,k)=(-1)^{(i+k)/2+1}\xi^{(1-k)/2}\bar{\xi}^{(1+k)/2}|\xi|^{-2(3/2+is)}\psi^\Gamma(s,\eta,z,t,i,-k), \tag{8.4}$$

with $\eta:=-\bar{\xi}^{-1}$.

If we replace now in the orthogonality relation (7.14) the ψ^Γ by $P^\Gamma\psi^\Gamma$, we get on the right hand side of Eq. (7.14) the ξ and ξ' replaced by η and η', and k by $-k$. A factor $|\xi|^{-4}$ is likewise to be added to the right hand side, stemming from the coefficient in front of ψ^Γ in Eq. (8.4).

Contrary to Eq. (7.14) however, the δ-functions $\delta(\gamma^{-1}\eta-\eta')$ cannot be neglected now for $\gamma\neq\text{id}$. This is so because the η, η' do not range any more in a fundamental domain of Γ in the complex plane, since the transformation $\xi\rightarrow-\bar{\xi}^{-1}$ is clearly not an element of Γ. Therefore the parity operator (8.4) is not norm preserving. [Accidentally $\xi\rightarrow-\bar{\xi}^{-1}$ can still be a symmetry, mapping a fundamental domain onto another. But we can take any point $\pi^\Gamma(z_0,t_0)$, $(z_0,t_0)\in H^3$, as the fixed point of a reflection, cf. the Remark after Eq. (5.4), and for almost all of these reflections the norm will not be preserved.]

On the other hand it is evident that C^Γ, T^Γ given by Eqs. (5.10) and (5.11) with ψ replaced by ψ^Γ stay antiunitary.

IX. CONCLUDING REMARKS

In Sec. VII we constructed unitary spinor representations of discrete subgroups of $SL(2,\mathbb{C})/\{\pm\text{id}\}$, which appear as the covering groups of the spacelike slices. These representations are needed to define periodic boundary conditions, cf. Eq. (7.5), for the Dirac equation on the fundamental polyhedron which represents the spacelike slices in the covering space. In this way we defined spinor fields and derived the spectral resolution by periodizing H^3 eigenfunctions. The existence of unitary spinor representations, cf. Eq. (7.3), depends on the existence of a character system satisfying Eq. (7.1). Such character systems exist for the covering groups of the manifolds described after Eq. (7.1), and we indicated an algorithmic generation of them. It would be very interesting to know under which conditions such character systems exist for more general Kleinian groups, cf. the Remark after Eq. (7.1). Some results in this direction can be found in Ref. 39 (pp. 356–358), and Ref. 30 (pp. 331–337).

There is a natural and straightforward way to construct space reflections on the multiply connected spacelike slices, just by combining a H^3 reflection with the universal covering projection, cf. Sec. VIII. Such reflections do not preserve the Riemannian volume, because a reflected domain can overlap with itself. This leads to self-interference effects, and to the violation of the unitarity of the parity operator. For neutrinos, cf. the end of Sec. V, *CP* is likewise a broken symmetry in a multiply connected universe.

In fact, self-interference of a wave packet can happen quite easily in the cosmologies studied here. Though the three-space is infinite, there may exist geodesic loops of quite a small size. It is not difficult to find simple examples for that. In Ref. 9 we discuss an open hyperbolic three-manifold that is topologically a solid torus, the product of a finite interval and an annulus. There exists a unique geodesic loop whose length l can be used to parametrize a path in the deformation space of the manifold. In other words, for every choice of $l, 0<l<\infty$ there exists a hyperbolic metric of sectional curvature -1 on the topological manifold, that gives rise to a loop of this size. A wave packet, that disperses in the vicinity of a tiny loop will easily start to interfere with itself. If one associates particles with topological excitations, one can start to speculate if the observed *CP* violation can be understood as a topological interference phenomenon. The difficulty is of course to make this quantitative.

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APPENDIX: CONVENTIONS AND CONNECTIONS

In this appendix we explain our notation concerning the metric, the Dirac matrices, and the covariant derivation of spinors. The coefficients Γ_μ of the spinor connection are calculated.

We use the Robertson–Walker (RW-) line element

$$ds^2=-c^2\,d\tau^2+a^2(\tau)d\sigma^2, \tag{A1}$$

with

$$d\sigma^2=R^2t^{-2}(|dz|^2+dt^2), \tag{A2}$$

and $z:=x^1+ix^2$, on the Poincaré half-space H^3, see, e.g., Ref. 24. We also use, especially if we carry out summations, x^0, x^3 instead of τ, t. The coefficients of the RW-metric $g_{\mu\nu}$ corresponding to ds^2 are

$$g_{00}=-c^2,\quad g_{ik}=a^2(\tau)R^2t^{-2}\delta_{ik},\quad g_{0i}=0. \tag{A3}$$

Latin indices run from 1 to 3, Greek from 0 to 3. With $\gamma_{ij}:=g_{ij}$ we denote the metric on the spacelike slices, corresponding to $a^2(\tau)d\sigma^2$.

Using the table for diagonal metrics in Ref. 40, we calculate easily the three-indices $\Gamma^\lambda_{\mu\nu}$ for the metric (A3)

$$\begin{gathered}\Gamma^1_{13}=\Gamma^2_{23}=\Gamma^3_{33}=-\Gamma^3_{11}=-\Gamma^3_{22}=-t^{-1},\\ \Gamma^1_{10}=\Gamma^2_{20}=\Gamma^3_{30}=\dot a/a,\\ \Gamma^0_{11}=\Gamma^0_{22}=\Gamma^0_{33}=R^2c^{-2}a\dot a t^{-2};\end{gathered} \tag{A4}$$

all other three-indices are either zero or can be obtained by interchanging the lower indices.

We use the following standard representation of the Dirac matrices $\tilde\gamma^\mu$ in Minkowski space, $\eta_{\mu\nu}:=\mathrm{diag}(-c^2,1,1,1)$:

$$\begin{gathered}\tilde\gamma^0:=\frac{-i}{c}\begin{pmatrix}\mathrm{id} & 0\\ 0 & -\mathrm{id}\end{pmatrix},\quad \tilde\gamma^k:=\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix},\\ \sigma_1:=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2:=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3:=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\end{gathered} \tag{A5}$$

We have

$$\tilde\gamma_\nu=\eta_{\nu\mu}\tilde\gamma^\mu,\quad \tilde\gamma_\nu\tilde\gamma_\mu+\tilde\gamma_\mu\tilde\gamma_\nu=2\eta_{\nu\mu},\quad \tilde\gamma_0^\dagger=-\tilde\gamma_0,\quad \tilde\gamma_k^\dagger=\tilde\gamma_k; \tag{A6}$$

the † denotes transposition and complex conjugation.

With respect to the metric (A3) we define the standard representation γ^μ of the Dirac matrices as

$$\gamma_0 := \tilde{\gamma}_0, \quad \gamma_k := \sqrt{g_{33}}\,\tilde{\gamma}_k. \tag{A7}$$

We have again Eq. (A6) with $\tilde{\gamma}^\mu$ replaced by γ^μ, and $\eta_{\mu\nu}$ by $g_{\mu\nu}$.

To calculate the connection symbols Γ_μ of the spinor connection we follow Ref. 20. At first we find a tetrad $b^\lambda_\mu(x)$ and its inverse $a^\lambda_\mu(x)$, so that

$$\eta_{\lambda\kappa} b^\lambda_\mu b^\kappa_\nu = g_{\mu\nu}, \quad a^\mu_\kappa b^\kappa_\nu = \delta^\mu_\nu. \tag{A8}$$

Then we calculate the connection symbols via the following procedure:

$$\begin{gathered} \Gamma_\kappa := \tfrac{1}{4} B_{\mu\nu\kappa}\, s^{\mu\nu} + A_\kappa \,\mathrm{id}, \quad s^{\mu\nu} := \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \\ B_{\mu\nu\kappa} := g_{\mu\lambda}(C^\lambda_{\nu\kappa} - \Gamma^\lambda_{\nu\kappa}), \quad C^\lambda_{\nu\kappa} := \frac{\partial b^\alpha_\nu}{\partial x^\kappa}\, a^\lambda_\alpha, \end{gathered} \tag{A9}$$

where $A_\kappa(x)$ denotes here an arbitrary vector field, the electromagnetic potential, and id the four by four unit matrix. The Γ_μ satisfy

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\kappa_{\nu\mu}\gamma_\kappa + \Gamma_\nu\gamma_\mu - \gamma_\mu\Gamma_\nu \tag{A10}$$

and Eq. (A9) is the most general solution of Eq. (A10).

We choose as a solution of Eq. (A8)

$$b^0_0 = a^0_0 = 1, \quad b^i_i = 1/a^i_i = \sqrt{g_{33}} \tag{A11}$$

and off-diagonal elements zero. We obtain

$$\begin{gathered} \Gamma_0 = A_0\,\mathrm{id}, \quad \Gamma_1 = \frac{1}{2}\left(-c^{-2}\frac{R}{t}\,\dot{a}(\tau)\tilde{\gamma}_0\tilde{\gamma}_1 + \frac{1}{t}\,\tilde{\gamma}_1\tilde{\gamma}_3\right) + A_1\,\mathrm{id}, \\ \Gamma_2 = \frac{1}{2}\left(-c^{-2}\frac{R}{t}\,\dot{a}(\tau)\tilde{\gamma}_0\tilde{\gamma}_2 + \frac{1}{t}\,\tilde{\gamma}_2\tilde{\gamma}_3\right) + A_2\,\mathrm{id}, \quad \Gamma_3 = -\frac{1}{2}\,c^{-2}\frac{R}{t}\,\dot{a}(\tau)\tilde{\gamma}_0\tilde{\gamma}_3 + A_3\,\mathrm{id}. \end{gathered} \tag{A12}$$

In this article we assume throughout $A_\kappa(x) = 0$. Finally we note

$$\begin{gathered} \gamma^1\Gamma_1 = \gamma^2\Gamma_2 = \frac{1}{2}\left(c^{-2}\frac{\dot{a}}{a}\,\tilde{\gamma}_0 + \frac{1}{Ra}\,\tilde{\gamma}_3\right), \quad \gamma^3\Gamma_3 = \frac{1}{2}\,c^{-2}\frac{\dot{a}}{a}\,\tilde{\gamma}_0, \\ \gamma^\lambda\Gamma_\lambda = \frac{3}{2}\,c^{-2}\frac{\dot{a}}{a}\,\tilde{\gamma}_0 + \frac{1}{Ra}\,\tilde{\gamma}_3. \end{gathered} \tag{A13}$$

The covariant derivatives of a spinor and its adjoint, $\bar{\psi} := \psi^\dagger\gamma^0$, are

$$\nabla_\lambda\psi := \frac{\partial\psi}{\partial x^\lambda} - \Gamma_\lambda\psi, \quad \nabla_\lambda\bar{\psi} := \frac{\partial\bar{\psi}}{\partial x^\lambda} + \bar{\psi}\Gamma_\lambda; \tag{A14}$$

an exposition of spinorial covariant differentiation can be found in Ref. 20, see also Ref. 21 and the literature cited therein.

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