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COSMIC TACHYON BACKGROUND RADIATION

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The equilibrium statistical mechanics of a background radiation of superluminal particles is investigated, based on a vectorial wave equation for tachyons of the Proca type. The partition function, the spectral energy density, and the various thermodynamic variables of an ideal Bose gas of tachyons in an open Robertson–Walker cosmology are derived. The negative mass square in the wave equation changes the frequency scaling in the Rayleigh–Jeans law, and there are also significant changes in the low temperature regime as compared to the microwave background, in particular in the caloric and thermal equations of state.

1. Introduction

We study the possibility of a cosmic background radiation of superluminal particles (tachyons) in an open Robertson–Walker (RW) cosmology with negatively curved three-space. Tachyons are regarded as the eigenmodes of a real vector field with negative mass square. Like the electromagnetic field, the tachyon field is conformally coupled to the background geometry, so that the frequencies of the spectral elementary waves scale inversely proportional to the curvature radius of the cosmic three-space. This allows us, despite the time variation of the background geometry, to use the thermodynamic equilibrium formalism and to scale the time dependence of the eigenmodes into the temperature variable, which becomes in this way a function of cosmic time.

It is crucial to consider superluminal signals in a cosmological context, because an obvious causality problem arising in Minkowski space whenever events are connected by superluminal signals[1] can readily be settled by invoking the galactic background as the universal frame of reference for all uniformly moving observers. This does not need any specific assumptions on the physical nature of the signal, i.e. on its interaction with matter, as suggested in Ref. 1. In fact, the causality

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principle is not based on physical concepts like energy, just the reverse holds. Causality is a category of our thinking, it is not a physical principle, nor is it a matter of mathematical logic, that can be altered or generalized whenever convenient. It is not advisable to specify cause and effect as physical processes in this principle, since physical processes are themselves described, usually tacitly, on the basis of the causality principle. Of course, causes and effects ultimately turn out to be physical effects, they can be characterized by energy transfer, but this description itself implies the causality principle. In Ref. 1, an attempt was made to generalize the causality principle and to distinguish and define cause and effect by the emission and absorption of positive energy, which amounts to the introduction of an antiparticle concept for classical superluminal point particles. Effects may then precede their causes. I do not think it permissible to modify the causality principle by means of physical concepts which are themselves based on this principle. The Newton law, for instance, cannot stand without the causality principle, nor can the energy conservation law, and not even probabilistic theories like quantum mechanics. The causality principle just asserts that every effect has a cause, that this cause precedes the effect, and that all observers can come to the same conclusion on what is cause and effect. In Minkowski space, time does not have an absolute meaning, and this gives rise to confusion concerning the time order of events connected by superluminal signals; either the second or the third condition of the causality principle has to be dropped if one considers superluminal signals in Minkowski space. However, the time order of cause and effect is determined by the absolute cosmic time which defines the expansion of the galactic background; if a signal moves from one location to another in cosmic time, it is clear what is meant by cause and effect, even if the time order of cause and effect may appear different in the proper time of locally geodesic observers. The galaxy background with its nearly perfectly Planckian microwave spectrum provides a unique reference frame and an unambiguous cosmic time order. Every observer can determine his motion relative to the microwave radiation, and so relate his individual rest frame to the comoving galaxy frame. In particular, he can compare the time order of events in his proper time to the universal cosmic time order. All observers can in this way reach identical conclusions on the causality of events connected by superluminal signals.

In Sec. 2, the classical mechanics of tachyons in the comoving galactic reference frame is discussed. We introduce a scalar wave equation for tachyons designed in a way that the semiclassical approximation is exact.[2] This real scalar field is extended to a vector field so that the phase of the spectral waves remains unchanged and the wave propagation is transversal; we arrive at a Proca equation[3] with negative mass square. In Sec. 3, the partition function and the spectral energy density of a free Bose gas of tachyons are calculated. The high and low temperature limits of internal energy, entropy, and pressure, as well as the heat capacities of the background radiation are determined. The negative mass square in the wave equation changes the frequency scaling in the low frequency limit, and there are also significant changes in the low temperature regime, in particular in the entropy per particle and the caloric and thermal equations of state. In Sec. 4 we comment on the interaction of tachyons with matter in this field theory and present our conclusions.

2. The Wave Equation for Tachyons

The RW cosmology is defined in comoving coordinates by the line element $d\lambda^2 = -d\tau^2 + a^2(\tau)d\sigma^2$, $a(\tau_0) = 1$. We use as coordinate representation of hyperbolic space the Poincaré half-space H^3, defined by the metric $d\sigma^2 = u^{-2}(du^2 + |d\xi|^2)$, with Cartesian coordinates u, ξ_1, ξ_2; $u > 0$, and $\xi = \xi_1 + i\xi_2 \cdot d\sigma^2$ induces constant negative curvature -1 on this half-space. The invariance group of H^3 is $\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$; see Ref. 4 for the group action on H^3. Tachyonic world-lines are defined by a Hamilton–Jacobi equation with negative mass square, $g^{\mu\nu} S_{,\mu} S_{,\nu} = \mu^2 \cdot [\mu^2 > 0$ in our notation, with $g_{\mu\nu}$ given by the RW line element $d\lambda^2$.] This is equivalent to the action principle $S = \int L d\lambda$, with $L(\lambda) = -\mu\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$. If we determine the world-lines along the u-semiaxis of H^3, then all other trajectories can be obtained by the mentioned symmetry transformations of H^3. The Lagrange equations result in $d\log u = s a^{-1}(s^2 - a^2\mu^2)^{-1/2} d\tau$ for tachyon trajectories along the u-semiaxis. (s is an integration constant.) From now on we assume that the tachyon mass μ is a scalar, varying in cosmic time, inversely proportional to the curvature radius of the three-space, $\mu = m/a(\tau)$, $m > 0$. The reason for this is that the wave equation for tachyons is conformally coupled, which requires a tachyon mass depending on cosmic time, see the discussion following Eq. (4), and the Remarks after Eq. (15). We obtain for the tachyonic world-lines

$$u(\tau) = \exp\left(\delta(s) \int a^{-1} d\tau\right), \qquad \delta(s) := \frac{s}{\sqrt{s^2 - m^2}}, \tag{1}$$

with $|s| > m$, corresponding to the action

$$S = -\sqrt{s^2 - m^2} \int a^{-1} d\tau + s \log u \,. \tag{2}$$

Tachyonic energy and momentum along the u-semiaxis read[5,6]

$$E = \frac{\mu}{\sqrt{|\mathbf{v}|^2 - 1}} = \frac{1}{a(\tau)}\sqrt{s^2 - m^2}\,, \qquad |\mathbf{v}| = |\delta(s)|\,, \tag{3}$$

$$\mathbf{p} = \frac{\mu \mathbf{v}}{\sqrt{|\mathbf{v}|^2 - 1}} = \frac{su}{a^2(\tau)}\,, \qquad |\mathbf{p}| = \frac{|s|}{a(\tau)}\,. \tag{4}$$

In the case of photons, the eikonal approximation is exact in RW cosmologies, since the electromagnetic potential is conformally coupled to the background geometry, which means here that the frequencies scale with the inverse of the expansion factor. This property we also want to retain for particles with negative mass square. We consider a real scalar field defined by the Lagrangian

$$L = -\frac{1}{2}\left[\psi_{,\nu}\psi^{,\nu} + \left(\frac{1}{6}\hat{R} - \mu^2\right)\psi^2\right], \qquad \frac{1}{6}\hat{R} := -\frac{1}{a^2} + \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\,. \tag{5}$$

The insertion of the Ricci scalar and the scaling of mass with the inverse of the expansion factor as indicated above guarantees the conformal coupling of the wave equation,

$$\psi_{,\mu}^{\;;\mu} - (\hat{R}/6 - \mu^2)\psi = 0\,. \tag{6}$$

Let us at first consider wave fields independent of the ξ-coordinate of H^3. The separation ansatz $\psi(\tau) = \chi(\tau) a^{-3/2} u^{1-is}$ gives

$$\frac{d^2\chi}{d\tau^2} + \left(\frac{1}{a^2}s^2 - \mu^2 - \frac{1}{2}\frac{\ddot{a}}{a} + \frac{1}{4}\frac{\dot{a}^2}{a^2}\right)\chi = 0\,. \tag{7}$$

As $\mu = m/a(\tau)$, we obtain the fundamental solutions

$$\chi = \sqrt{a}\exp\left(\pm i\sqrt{s^2 - m^2}\int a^{-1} d\tau\right). \tag{8}$$

Thus, the wave equation (6) is solved by

$$\psi(\tau, u; s) = a^{-1}(\tau) u \exp(-iS) \tag{9}$$

(real and imaginary part, we use in the following complex notation for convenience); the phase is identical with the action (2). The energy of the spectral modes (9) is defined by $E(s) = -\partial S/\partial\tau$, which coincides with the classical energy (3), of course.

Remark. We have put $c = \hbar = R = 1$. R is the curvature radius at the present epoch τ_0. To restore the natural units, we replace in the action m by mc, and multiply S by R. In the wave equation (6), we replace m by $mc/\hbar$, and in the formula (5) for the Ricci scalar $\hat{R}$ we multiply the first term by R^{-2}, and the second and third by c^{-2}. The curvature radius of the cosmic three-space is $Ra(\tau)$. Frequency and wave vector are related to energy and momentum as $\omega = E/\hbar$, and $\mathbf{k} = \mathbf{p}/\hbar$, with $\mathbf{p} = g^{uu}\partial S/\partial u$, cf. Eq. (4), since the semiclassical approximation is exact. For group and phase velocity we obtain $|\mathbf{v}_{\rm gr}| = c^2|\mathbf{v}_{\rm ph}|^{-1} = c|\delta(s)|$. Group and particle velocity coincide, and can be made arbitrarily large by the choice of the integration parameter (spectral variable) s. For the rest of this article, ω denotes a complex spectral variable.

A complete set of elementary waves $\psi(\tau, u, \xi; s, \omega)$ is generated by applying certain symmetry transformations of H^3 onto the modes (9),

$$(u, \xi) \to (|\xi - \omega|^2 + u^2)^{-1}(u, \overline{\xi - \omega})\,, \tag{10}$$

$\omega \in \mathbb{C}$ cf. Ref. 4. This simply means to substitute into Eq. (9) the Poisson kernel,

$$u \to P(u, \xi; \omega) = \frac{u}{|\xi - \omega|^2 + u^2}\,. \tag{11}$$

In particular,

$$S(\tau, u, \xi; s, \omega) = -\sqrt{s^2 - m^2}\int a^{-1} d\tau + s\log P(u, \xi; \omega)\,, \tag{12}$$

is the general solution of the Hamilton–Jacobi equation, as well as the phase of the spectral waves. The modes (9), with (11) and (12) substituted, constitute a complete set of eigenmodes of the wave equation (6).

The vectorial extension of this wave equation is based on the Lagrangian

$$L = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\mu^2 A_\alpha A^\alpha + A_\alpha j^\alpha \,, \tag{13}$$

with $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$ and $\mu = m/a(\tau)$, $m > 0$, which leads to the Proca equation

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{\alpha\beta})}{\partial x^\beta} - \mu^2 A^\alpha = j^\alpha \tag{14}$$

with negative mass square for a real vector potential A_α. Differentiating (14), and using the skew symmetry of $F^{\alpha\beta}$, we obtain

$$A^\alpha{}_{;\alpha} + \frac{\mu^2{}_{,\alpha}}{\mu^2}A^\alpha = -\frac{1}{\mu^2}j^\alpha{}_{;\alpha} \,. \tag{15}$$

If the current is conserved, this evidently is the analogue to the Lorentz condition, but unlike the Lorentz condition, Eq. (15) is a consequence of the field equations, and the mass term also breaks the gauge symmetry.

To solve the wave equation (14) with $j^\alpha = 0$, i.e. to find its spectral resolution, we try the separation ansatz $A_0 = A_1 = A_3 = 0$, $A_2 = \varphi(\tau)u^{is}$, so that $F^{02} = -a^{-2}\varphi'(\tau)u^{2+is}$ and $F^{21} = -isa^{-4}\varphi(\tau)u^{3+is}$, all other components of $F^{\alpha\beta}$ vanish. In this way the $\alpha = 0, 1, 3$ components of the field equation are identically satisfied, and the $\alpha = 2$ component leads, if we write $\varphi = a^{-1/2}\chi$, just to Eq. (7). Accordingly, the phase of the spectral modes coincides with that of the scalar wave fields (9), namely with the action (2). A second independent set of transversal modes is obtained by interchanging A_2 and A_3 in the above ansatz. Finally, a complete set of eigenmodes is generated by applying the symmetry transformation (10) to these two linearly polarized sets. The phase of these modes is the action (12).

Remarks. By restoring the natural units in the Lagrangian (13), we find $\mu = mc/(\hbar a(\tau))$. Energy and momentum are proportional to frequency and wave vector via the de Broglie/Einstein relation, since the phase of the spectral modes coincides with the classical action. There are no antitachyons;[1] the choice of the Lagrangian (13) as an extension of classical electrodynamics rather than of subluminal massive field theories is also suggested by the fact that tachyons do not have a rest mass. The variation of the tachyon mass in cosmic time can also be interpreted in terms of a cosmic ether,[7–9] and strongly reminds us on the varying fundamental constants of Eddington, Milne, and Dirac. The Lagrangian (13) determines a well defined interaction mechanism of the tachyon field with subluminal particles, analogous to electrodynamics; the current in the wave equation (14) is assumed to be structured as in electrodynamics, cf. Sec. 4.

3. Equilibrium Thermodynamics of a Free Tachyon Gas

Let us turn to the partition function of the cosmic tachyon background radiation, calculated via the usual box quantization procedure[10,11] of a free Bose gas,

$$\begin{aligned} \log Z &= 2\log \sum_{(n_p)=0}^{\infty} \exp\left(-\beta \sum_p h\nu(p) n_p\right) \\ &= -2\sum_p \log[1-\exp(-\beta h\nu(p))]\,, \end{aligned} \tag{16}$$

$$\beta^{-1} := kT(\tau) = kT_0/a(\tau)\,.$$

The chemical potential is zero, which follows from an equilibrium condition to be imposed on the free energy, see the discussion following Eq. (23). The index p runs over a discrete set of values, $p = hL^{-1}(k_1, k_2, k_3)$, $k_i \in \mathbb{Z}$, L the box size, and (n_p) is a multi-index labeled by p. Euclidean box quantization with periodic boundary conditions also applies here, if the box size is much smaller than the curvature radius of the universe. The space component of the spectral modes (9)–(12) of the scalar wave equation (6) can be identified in this limit with the Euclidean exponentials $\exp(i\mathbf{kx})$, if we use as coordinate representation of the cosmic three-space Poincaré's ball-model for the hyperbolic three-space,[4,9] also compare Sec. 4. The proper frequencies of the Proca equation coincide with those of the scalar equation, as pointed out after Eq. (15), and the factor of two in Eq. (16) accounts for the two independent transversal states. The frequencies conformally scale with the inverse of the expansion factor, cf. Eq. (3) and the Remark following Eq. (9), and this time dependence can be scaled into the temperature variable.[12] (If the tachyon mass were constant, we would have to appeal to the adiabatic time variation of the expansion factor, in order to justify an equilibrium distribution.) In the thermodynamic limit, the summation over the lattice points p is replaced by an integral, and we arrive at

$$\log Z = -\frac{2V}{h^3}\int_{|p|>mc} d^3p \log[1-\exp(-\beta h\nu(p))]\,, \quad h\nu(p) = c\sqrt{|p|^2-(mc)^2}\,. \tag{17}$$

If we put $m = 0$, this coincides with the partition function of the photon gas, of course. The spectral energy density thus reads

$$\rho(\nu)d\nu = \frac{8\pi V}{h^3}\frac{h\nu(p)|p|^2 d|p|}{\exp[h\nu(p)/(kT(\tau))]-1}\,, \tag{18}$$

and the internal energy is obtained as

$$\begin{aligned} U &= \int_0^\infty \rho(\nu)d\nu = 8\pi m^4 c^5 h^{-3} V\hat{U}(\alpha)\,, \\ \hat{U}(\alpha) &:= \int_0^\infty \frac{x^2\sqrt{1+x^2}dx}{\exp(\alpha x)-1}\,, \qquad \alpha := \frac{mc^2}{kT(\tau)}\,. \end{aligned} \tag{19}$$

In the low and high temperature limits we find

$$U(T \to 0) \sim 16\pi\zeta(3)\frac{mk^3}{ch^3}VT^3\,, \qquad U(T \to \infty) \sim \frac{8\pi^5}{15}\frac{k^4}{h^3c^3}VT^4\,. \tag{20}$$

The partition function (17) we may write as

$$\log Z = -\frac{8\pi V}{h^3}\int_{mc}^{\infty}|p|^2 d|p| \log\left[1 - \exp\left(-\alpha\sqrt{\frac{|p|^2}{(mc)^2} - 1}\right)\right], \tag{21}$$

and the free energy as

$$\begin{aligned} F &= -\frac{8\pi}{3}\frac{m^4c^5}{h^3}V\hat{F}(\alpha)\,, \\ \hat{F}(\alpha) &:= -\frac{3}{\alpha}\int_0^{\infty} x\sqrt{x^2+1}\log(1-e^{-\alpha x})dx \\ &= \int_0^{\infty}\frac{(x^2+1)^{3/2}-1}{e^{\alpha x}-1}dx\,. \end{aligned} \tag{22}$$

Internal energy and entropy are related to $\hat{F}(\alpha)$ via

$$\hat{U}(\alpha) = -\frac{1}{3}\frac{\partial(\alpha\hat{F}(\alpha))}{\partial\alpha}\,, \qquad S = -\frac{8\pi}{3}\frac{km^3c^3}{h^3}V\alpha^2\frac{\partial\hat{F}(\alpha)}{\partial\alpha}\,. \tag{23}$$

The relevant quantities are of course $u = U/V$ and $s = S/V$, but we write in the following the volume factor and capital letters to avoid notational confusion. As $\partial F(T,V,N)/\partial N = \mu = 0$, the pressure reads $P = -F/V$. The condition $\partial F/\partial N = 0$ is necessary to impose. Tachyons, like photons, are not interacting with each other. Thus, for equilibrium to be reached, we must assume interaction with subluminal matter, absorption and emission processes. Therefore N cannot be kept constant, and then $\partial F/\partial N = 0$ is a necessary extremal condition for equilibrium.[11] Accordingly, we have to put $\mu = 0$ in the partition function. At any rate, the tachyon number N is not an independent variable, and can be calculated as a function of temperature,

$$\begin{aligned} N &= \frac{8\pi V}{h^3}\int_{mc}^{\infty}\frac{|p|^2 d|p|}{\exp\left[\alpha\sqrt{|p|^2/(mc)^2-1}\right]-1} \\ &= \int_0^{\infty} n(\nu)d\nu = 8\pi\frac{m^3c^3}{h^3}V\hat{N}(\alpha)\,, \\ \hat{N}(\alpha) &:= \int_0^{\infty}\frac{\sqrt{x^2+1}}{e^{\alpha x}-1}x\,dx\,. \end{aligned} \tag{24}$$

As the chemical potential vanishes, the enthalpy just reads $H = TS$. Thus we have for the heat capacities $c_v = \partial U/\partial T$ and $c_p = \partial H/\partial T = c_v + S$. In the low temperature regime we obtain the asymptotic relations

$$\begin{aligned} N &\sim \frac{4}{3}\pi^3\frac{mk^2}{ch^3}VT^2\,, & PV &\sim \frac{1}{2}U \sim \frac{6\zeta(3)}{\pi^2}NkT\,, \\ S &\sim \frac{1}{2}c_v \sim \frac{3}{2}\frac{U}{T} \sim \frac{18\zeta(3)}{\pi^2}Nk\,, & \frac{c_p}{c_v} &\sim \frac{3}{2}\,, \end{aligned} \tag{25}$$

with $\zeta(3)/\pi^2 \approx 0.122$. The entropy and the thermal equation of state are in the indicated leading order independent of the tachyon mass, but not so the caloric equation (20).

The high temperature expansions are convergent, and we find in leading order

$$\begin{aligned} N &\sim 16\pi\zeta(3)\frac{k^3}{h^3c^3}VT^3\,, & PV &\sim \frac{1}{3}U \sim \frac{\pi^4}{90\zeta(3)}NkT\,, \\ S &\sim \frac{1}{3}c_v \sim \frac{4}{3}\frac{U}{T} \sim \frac{2\pi^4}{45\zeta(3)}Nk\,, & \frac{c_p}{c_v} &\sim \frac{4}{3}\,. \end{aligned} \tag{26}$$

The tachyon mass drops out in this limit, and relations (26) hold as equalities for the photon gas over the whole temperature range.[11]

Let us finally return to the spectral energy density (18),

$$\rho(\nu) = \frac{8\pi h}{c^3}\frac{\nu^2\sqrt{\nu^2 + m^2c^4/h^2}}{\exp(h\nu/(kT)) - 1}\,. \tag{27}$$

(We now drop the volume factor.) In the high frequency limit, Wien's radiation law is still recovered, but not so the Rayleigh–Jeans law in the low frequency regime, because $\rho(\nu \to 0) \sim 8\pi c^{-1}h^{-1}m\nu kT$. (In the massless case, we have for low frequencies $\rho \sim 8\pi c^{-3}\nu^2 kT$, independent of the Planck constant, and quadratic in the frequency.) Since the spectral particle density relates to the energy density as $n(\nu) = \rho(\nu)/(h\nu)$, cf. Eq. (24), $n(\nu \to 0)$ approaches a finite value depending on the tachyon mass, whereas the photon density vanishes in this limit. There is of course no Bose condensation possible, due to the lack of a chemical potential and a rest mass.

4. Conclusion

I conclude by comparing the theory of superluminal wave propagation suggested in this paper with the "standard approach," and I also shortly discuss the static tachyon potential and how it affects atomic energy levels. Tachyons are usually introduced as an extension of the relativistic particle concept, as particles with negative mass square, or, if one prefers, imaginary mass.[13,14] In the relativistic Hamilton–Jacobi equation, this just means to assume $m^2 < 0$ without further alterations. If tachyons are supposed to carry electric charge, then their coupling to the electromagnetic potential is effected by minimal substitution, $S_{,\mu} \to S_{,\mu} - eA_\mu$, and the Lagrangian for tachyons can be written in the same way as for subluminal particles, $L = -\sqrt{-m^2\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + eA_\mu\dot{x}^\mu$, $\mathrm{diag}(\eta_{\mu\nu}) = (-1, 1, 1, 1)$, but with $m^2 < 0$. Sommerfeld's pre-relativistic study of superluminal motion[15,16] was aimed at accelerating electrons beyond the speed of light by means of electromagnetic fields, but otherwise his view of tachyons as particles coupled to the electromagnetic field in the usual way was taken over by modern authors. By contrast, in this article superluminal signals are themselves modeled as wave fields, by a Proca equation with negative mass square, very contrary to the prevailing view of tachyons as charged point particles with imaginary mass and zero spin. The superluminal wave modes of the tachyon field are coupled to a current of subluminal particles by minimal substitution. In short, in the above Lagrangian we regard the field A_μ as superluminal rather than the particle. Superluminal wave propagation, the spectral resolution of the wave equation (14), and the semiclassical approximation are further discussed in Ref. 17.

Finally, if one contemplates on the possibility of superluminal signals, one is faced with the choice of an interaction mechanism with subluminal matter. In Sec. 3, we had to assume some kind of interaction of the tachyon gas with matter for equilibrium to be reached. The field theory studied in this paper suggests an interaction mechanism of tachyons with subluminal matter in analogy to electrodynamics. As an example, we derive the potential of a static point source carrying tachyonic charge. Because of the spherical symmetry of the potential, we use as coordinate representation of the hyperbolic three-space the ball model of hyperbolic space, $d\sigma^2 = 4(1-|\mathbf{x}|^2)^{-2}d\mathbf{x}^2$, $|\mathbf{x}| < 1$, which is isometric to the line element of the hyperbolic half-space H^3 as defined at the beginning of Sec. 2, cf. Ref. 4. The potential of a static point source defined by the current $j^0 = q(-g)^{-1/2}\delta(\mathbf{x})$, $j^k = 0$, is readily calculated from the field equations (14). These equations reduce, by the ansatz $A_0 = a^{-1}f(r)$, $A_i = 0$, and $F_{0i} = -\varphi(\tau)f'(r)x_i/r$, to

$$\frac{1}{4}(1-r^2)^2\Delta_E f + \frac{1}{2}(1-r^2)rf' + m^2 f = \frac{q}{8}(1-r^2)^3\delta(\mathbf{x})\,, \tag{28}$$

with $\Delta_E f = r^{-2}d(r^2 f')/dr$. The fundamental solutions are

$$f_\pm(r) = r^{-1}(1-r^2)^{1\mp i\sqrt{m^2-1}}(1-r)^{\pm 2i\sqrt{m^2-1}}\,. \tag{29}$$

We assume $m^2 > 1$, which means a negative mass square in the notation of this paper (except in the Lagrangian at the beginning of this section). Thus, the potential reads as

$$\begin{aligned} A_0 &= -\frac{q}{4\pi aR}\frac{\cos(\kappa d(r)) + \lambda\sin(\kappa d(r))}{\sinh(a^{-1}R^{-1}d(r))}\,,\\ d(r) &:= Ra(\tau)\log\frac{1+r/R}{1-r/R}\,,\\ \kappa &:= \sqrt{\left(\frac{mc}{a\hbar}\right)^2 - \frac{1}{(Ra)^2}}\,; \end{aligned} \tag{30}$$

we have here restored the natural units. $d(r)$ is the radial distance function of the cosmic three-space (with line element $R^2a^2(\tau)d\sigma^2$). Both fundamental solutions (29) have the same exponential decay at infinity, i.e. for distances larger than the curvature radius; therefore one cannot select a special solution (defined by the integration constant λ) on the grounds of fastest decay, which is the usual argument to fix the potential in a field theory with nonnegative mass square. However, we can require the tachyon potential to be singularity free at $r = 0$, so that it admits a finite classical self-energy, which means to drop the cosine and to put $\lambda = 1$, cf. Ref. 18. For $r \to 0$, we find $A_0 \sim -q/(8\pi ar)$; this is the singularity required by Eq. (28), which reduces to the Poisson equation, $\Delta_E r^{-1} = -4\pi\delta(\mathbf{x})$, in this limit. We obtain the local Euclidean limit of the potential, if we identify $d(r)$ with the Euclidean distance and perform the limit $R \to \infty$ in (30). In the Hamilton–Jacobi equation, the tachyon potential is superposed on the Coulomb potential as a perturbation,

$$U(r) = \frac{\alpha}{r} + \frac{\beta}{r}(\cos(\kappa r) + \lambda \sin(\kappa r)) , \tag{31}$$

where $\alpha := e_1 e_2/(4\pi)$, $\beta := -q_1 q_2/(4\pi)$, and $\kappa = mc/\hbar > 0$. Here, $e_{1,2}$ and $q_{1,2}$ are the electric and tachyonic charges of source and particle, and m denotes the tachyon mass at the present epoch. We can study the effect of the tachyon potential on atomic energy levels by Bohr quantization, and estimate the tachyon mass by comparing high-precision measurements of the $1S - 2S_{1/2}$ and Ly-α_1 transitions in hydrogen and hydrogen-like ions with Lamb shift calculations.[18] We find $m \approx 3.06$ keV$/c^2$, corresponding to a Compton wave length of 6.45×10^{-9} cm. An estimate of the coupling constant of the tachyon potential (pure sin-potential with finite self-energy) is likewise obtained in this way, $\beta/\alpha \approx 9.3 \times 10^{-12}$ for hydrogen. As the tachyon radiation is in equilibrium with the microwave background at $T \approx 2.73$ K, we find $mc^2/(kT) \approx 1.3 \times 10^7$, and hence $U_{\mathrm{tach}}/U_{\mathrm{ph}} \approx 4.8 \times 10^6$ (energy densities of tachyon and photon background radiations). Quantitative estimates on atomic absorption and emission rates for tachyon radiation are given in Ref. 18; the high tachyonic energy desntity can only in part compensate the small ratio β/α of tachyonic and electric fine structure constants, which makes the tachyon background difficult to observe.

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