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White dwarf stars exceeding the Chandrasekhar mass limit

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HIGHLIGHTS

- A nearly degenerate electron plasma pervading an ionized high-density background medium is studied, as it occurs in stellar matter.
- The Dirac equation coupled to the permeability tensor of the medium leads to nonlinear electron dispersion in the ultra-relativistic regime.
- The quantized spectral density of a low-temperature electron gas in a dispersive medium is shown to be mechanically and thermally stable.
- The nonlinear electron dispersion affects the mass–radius relation of white dwarfs, whose mass can surpass the Chandrasekhar limit.
- White dwarf progenitors of super-Chandrasekhar mass Type Ia supernovae: estimates of their central mass density, incompressibility and speed of sound.

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ABSTRACT

The effect of nonlinear ultra-relativistic electron dispersion on the mass–radius relation of high-mass white dwarfs is studied. The dispersion is described by a permeability tensor in the Dirac equation, generated by the ionized high-density stellar matter, which constitutes the neutralizing background of the nearly degenerate electron plasma. The electron dispersion results in a stable mass–radius relation for high-mass white dwarfs, in contrast to a mass limit in the case of vacuum permeabilities. In the ultra-relativistic regime, the dispersion relation is a power law whose amplitude and scaling exponent is inferred from mass and radius estimates of two high-mass white dwarfs, Sirius B and LHS 4033. Evidence for the existence of super-Chandrasekhar mass white dwarfs is provided by several Type Ia supernovae (e.g., SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc), whose mass ejecta exceed the Chandrasekhar limit by up to a factor of two. The dispersive mass–radius relation is used to estimate the radii, central densities, Fermi temperatures, bulk and compression moduli and sound velocities of their white dwarf progenitors.

1. Introduction

The purpose of this paper is to study the effects of nonlinear electron dispersion in high-mass white dwarfs. We derive a stable mass–radius relation which remains valid above the Chandrasekhar mass limit of 1.44 solar masses due to nonlinear electron dispersion at ultra-relativistic energies. To this end, we determine the impact of dispersion on the thermodynamic variables of a nearly degenerate (high-density low-temperature) electron plasma. The quantized spectral density of the ultra-relativistic electrons is obtained by coupling the Dirac equation to the permeability tensor generated by the ionized stellar matter. The electronic dispersion relation defined by the permeabilities admits a power-law form in the ultra-relativistic

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regime, whose amplitude and scaling exponent can empirically be determined from mass and radius measurements of high-mass white dwarfs.

We calculate the entropy variable subject to electron dispersion and demonstrate thermodynamic stability, that is positive heat capacities and compressibilities. We derive the thermal equation of state of the electron plasma, which is polytropic in the totally degenerate ultra-relativistic regime, and then use the Lane–Emden equation for polytropes to derive the mass–radius relation for high-mass white dwarfs.

This mass–radius relation crucially depends on the amplitude and the scaling exponent of the electronic dispersion relation. In the case of vacuum permeabilities, the ultra-relativistic dispersion relation is linear, resulting in a mass limit instead of a mass–radius relation. In contrast, we treat the ionized stellar matter as a permeable medium pervaded by the electron gas, and infer the permeabilities from mass and radius estimates of high-mass white dwarfs. In this way, we can specify all parameters in the dispersive ultra-relativistic mass–radius relation.

The mass ejecta of several Type Ia (thermonuclear) supernovae (e.g., SN 2013cv [1], SN 2003fg [2], SN 2007if [3] and SN 2009dc [4]) substantially exceed the mass limit of 1.44 $M_\odot$ and suggest super-Chandrasekhar mass progenitors. Using estimates of the ejecta mass and applying the dispersive mass–radius relation, we derive radius and density estimates of their white dwarf progenitors. The electron dispersion is treated as a genuine nonlinear effect rather than a small perturbation of the linear vacuum dispersion relation, given that the mass of the white dwarf progenitor of supernova SN 2009dc exceeds the Chandrasekhar mass limit of 1.44 $M_\odot$ by almost a factor of two. The central density of the SN 2009dc progenitor reaches the neutron drip density, so that white dwarf masses much higher than 2.8 $M_\odot$ are not attainable.

In the following, we give an outline of this paper. In Section 2, we study relativistic fermionic spectral densities in a dispersive medium. The dispersion relation is derived from the Dirac equation coupled to an isotropic energy-dependent permeability tensor in analogy to electromagnetic theory. This changes the linear ultra-relativistic vacuum relation $E \sim p$ into a power law, $E(p) \sim m(p/m)^\eta/(\varepsilon_0\mu_0)$, where ε_0 and μ_0 are permeability amplitudes, m denotes the electron mass and η is a positive scaling exponent. The spectral decay of the Fermi distribution $\mathrm{d}\rho(p) \propto p^2\mathrm{d}p/(\mathrm{e}^{\alpha+\beta E(p)}+1)$ (where α and β are fugacity and temperature parameters defining the chemical potential $\mu = -\alpha/\beta$) is of Weibull type [5], the decay factor $\exp(-\beta E(p \to \infty))$ being a stretched ($\eta < 1$) or compressed ($\eta > 1$) exponential.

Weibull exponentials have been extensively applied in statistical modeling. A stretched (subexponential) Weibull factor appears as Kohlrausch function in anomalous diffusion and relaxation processes [6,7]; recent examples include diffusion in magnetic resonance imaging [8], relaxation of nanoparticles in liquids [9] and diminution processes in viscous media [10]. The tensile fracture probability of fiber bundles is modeled as subexponential Weibull density in Refs. [11,12]. Astrophysical applications of subexponential Weibull factors include asteroid fragmentation statistics [13], population decay statistics of satellite ejecta [14], cosmic ray statistics [15–18], and velocity distributions of planetary surface winds [19–21]. The population growth and epidemic models in Refs. [22,23] exemplify interdisciplinary applications of sub- and super-exponential (compressed) Weibull factors. Densities interpolating between Weibull exponentials and power laws have been used to model wealth distributions [24] and stock market volatility [25,26] as well as interevent times in human dynamics [27]. Network applications of Weibull densities are discussed in Refs. [28–31].

The Weibull decay of the above stated Fermi distribution is sub- or super-exponential for dispersion exponents $\eta < 1$ and $\eta > 1$, respectively, which affects the fugacity expansions discussed in Section 3, where we study the nearly degenerate ultra-relativistic quantum regime, subject to nonlinear electron dispersion. Starting with the integral representations of the thermodynamic variables derived in Section 2, we perform a high-density low-temperature fugacity expansion, obtaining the two leading orders of the electronic number count, internal energy, pressure and entropy in (α, β, V) parametrization.

In Section 4, we discuss the effect of the nonlinear dispersion relation on the mechanical and thermal stability of the electron gas in the nearly degenerate regime. By switching to the (N, β, V) representation of the energy, pressure and entropy variables, we derive the thermal equation of state, the isochoric and isobaric heat capacities $C_{V,P}$ and the isobaric expansion coefficient, as well as the isothermal and adiabatic compressibilities $\kappa_{T,S}$. We demonstrate, by explicit calculation, that the equilibrium stability conditions $\kappa_T > \kappa_S > 0$ and $C_P > C_V > 0$ are satisfied for positive scaling exponents η in the dispersion relation. We also obtain fugacity expansions of the adiabatic bulk modulus, the compression modulus (adiabatic incompressibility) and the speed of sound in the ionized background medium.

In Section 5, we study the effect of electron dispersion on the mass–radius relation of high-mass white dwarfs. We employ the thermal equation of state in the totally degenerate ultra-relativistic regime, $P \propto (N/V)^{1+\eta/3}$, where η is the scaling exponent of the electronic dispersion relation $E \sim m(p/m)^\eta/(\varepsilon_0\mu_0)$. As the thermal equation has a polytropic form, the stellar structure equations can be reduced to the Lane–Emden equation, which admits stable solutions for scaling exponents $\eta > 1$ and allows us to derive an explicit mass–radius relation for high-mass white dwarfs. This dispersive mass–radius relation depends on the scaling exponent η and the product $\varepsilon_0\mu_0$ of the permeability amplitudes in the electronic dispersion relation. These are two additional (fitting) parameters as compared with the linear ultra-relativistic dispersion relation $E \sim p$ in vacuum ($\varepsilon_0\mu_0 = \eta = 1$) which gives a mass limit instead of a mass–radius relation. The Lane–Emden equation does not define a mass limit for dispersion exponents $\eta > 1$, which opens the possibility of super-Chandrasekhar mass white dwarfs discussed in Section 6.

In Section 6.1, we use the mass–radius data of two high-mass white dwarfs, Sirius B [32] and LHS 4033 [33], and the dispersive mass–radius relation derived in Section 5 to infer the scaling exponent $\eta = 1.240$ and the amplitude $\varepsilon_0\mu_0 = 4.85/\mu_\mathrm{n}^{1+\eta/3}$ of the electronic dispersion relation. μ_n denotes the molecular weight per electron (nucleon–electron ratio, approximately $\mu_\mathrm{n} \approx 2$ for white dwarfs, unless neutronization by electron capture sets in, which increases μ_n, cf.

Section 6.3). The scaling exponent $\eta = 1.240$ is safely in the stability domain $\eta > 1$ of the Lane–Emden equation. In Section 6.1, we also derive the radii and central mass densities of the white dwarf progenitors of the super-Chandrasekhar mass supernovae SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc. Estimates of the central Fermi momentum and Fermi temperature of the progenitor stars are given in Section 6.2, and their bulk and compression moduli, sound velocity and gravitational surface potential are calculated in Section 6.3. In Section 7, we present our conclusions.

2. Fermionic spectral densities in a dispersive medium

2.1. Nonlinear ultra-relativistic dispersion relation

We start with the electronic Dirac equation coupled to a permeability tensor [34–36], and consider plane wave solutions, $\psi = \exp(\mathrm{i}p_\mu x^\mu)u(p)$, $p_\mu = (-E, \mathbf{p})$. The spinor $u(p)$ satisfies the Dirac equation in momentum space coupled to a dispersive isotropic permeability tensor $g^{\mu\nu}(p)$,

$$(\mathrm{i}\gamma_\mu g^{\mu\nu} p_\nu + m)u(p) = 0, \quad g^{00} = -\varepsilon(p), \quad g^{ik} = \frac{\delta^{ik}}{\mu(p)}, \tag{2.1}$$

with vanishing flanks $g^{0i} = 0$. The sign convention for the Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the Dirac matrices satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$, and m is the electron mass. By squaring the Dirac equation and using the plane-wave ansatz as stated above, we find the Klein–Gordon equation coupled to the squared permeability tensor $h^{\mu\nu} = g^{\mu\alpha}\eta_{\alpha\beta}g^{\beta\nu}$:

$$(h^{\mu\nu} p_\mu p_\nu + m^2)u(p) = 0, \quad h^{00} = -\varepsilon^2, \quad h^{ik} = \frac{\delta^{ik}}{\mu^2}, \tag{2.2}$$

and $h^{0i} = 0$, which defines the electronic dispersion relation

$$E(p) = \frac{\sqrt{p^2 + \mu^2(p)m^2}}{\varepsilon(p)\mu(p)}. \tag{2.3}$$

($\hbar = c = 1$.) We will focus on the ultra-relativistic limit, $p/m \gg 1$, and assume power-law asymptotics of the permeabilities, $\varepsilon \sim \varepsilon_0(p/m)^\chi$, $\mu \sim \mu_0(p/m)^\varphi$, with positive dimensionless amplitudes ε_0, μ_0 and real exponents χ and φ. For exponents $\varphi < 1$, the ultra-relativistic limit of the dispersion relation (2.3) reads

$$E = \frac{m}{\varepsilon_0\mu_0}(p/m)^\eta, \quad \eta = 1 - \chi - \varphi, \tag{2.4}$$

since the mass term $\mu^2(p)m^2$ in (2.3) drops out in leading order. (The electron mass in (2.4) is just a convenient scale parameter.) The group velocity $\eta E/p$ can approach zero ($0 < \eta < 1$) or become superluminal ($\eta > 1$) in the ultra-relativistic limit; the electromagnetic counterpart is 'slow' or 'fast' light in highly dispersive media [37–39]. If $\eta < 0$, the group velocity is negative, pointing in the opposite direction of the energy transfer [40]. We will use the shortcut $E = p^\eta/a^\eta$, with amplitude $a^\eta = m^{\eta-1}\varepsilon_0\mu_0$, and also restrict the exponent η to be positive. In the case of vacuum permeabilities, $\chi = \varphi = 0$, $\eta = 1$, $\varepsilon_0 = \mu_0 = 1$, the dispersion relation (2.4) is linear. The permeabilities can be reparametrized by energy via (2.4), but we will use a momentum rather than energy parametrization of the spectral density, see (2.5).

In the non-relativistic regime, $p/m \ll 1$, we assume constant positive permeabilities, $\varepsilon \sim \varepsilon_{\mathrm{nr}}$, $\mu \sim \mu_{\mathrm{nr}}$ instead of power laws and find, by expanding (2.3), the dispersion relation $E \sim m/\varepsilon_{\mathrm{nr}} + p^2/(2\varepsilon_{\mathrm{nr}}\mu_{\mathrm{nr}}^2 m)$. This resembles the dispersion relation in electronic band theory, where $\varepsilon_{\mathrm{nr}}\mu_{\mathrm{nr}}^2 m$ is the effective mass and $m/\varepsilon_{\mathrm{nr}}$ the band gap. In band theory, the effective mass is generated by adding a perturbative Bloch potential to the free electronic Lagrangian, whereas the permeability tensor in (2.1) and (2.2) affects the kinetic part of the Lagrangian [34,35]. In this paper, we will study the ultra-relativistic regime, $p/m \gg 1$, admitting the dispersion relation (2.4).

2.2. Quantized thermodynamic variables

We will study an electron gas at low temperature and high density, defined by the spectral number density

$$\mathrm{d}\rho(p) = \frac{4\pi s}{(2\pi)^3} \frac{p^2 \mathrm{d}p}{\mathrm{e}^{\alpha + \beta E(p)} + 1}, \tag{2.5}$$

where $E(p)$ is the ultra-relativistic dispersion relation (2.4), $s = 2$ is the electronic spin degeneracy, $f = \mathrm{e}^{-\alpha}$ the fugacity, and $\beta = 1/(k_{\mathrm{B}}T)$ the temperature parameter. We will mostly put $\hbar = c = k_{\mathrm{B}} = 1$ and use the shortcut $E = p^\eta/a^\eta$ for the dispersion relation (2.4), with power-law exponent $\eta > 0$ and amplitude $a^\eta = m^{\eta-1}\varepsilon_0\mu_0$, where ε_0 and μ_0 are positive permeability constants and m is the electron mass. We will also write $\beta E(p) = \hat{\beta} p^\eta$, with the rescaled temperature parameter $\hat{\beta} = \beta/a^\eta$. The classical spectral density is recovered if the fugacity is small, $\mathrm{d}\rho \sim 4\pi s(2\pi)^{-3}\mathrm{e}^{-\alpha-\beta E(p)}p^2\mathrm{d}p$. This is also the high-energy limit of (2.5). The Weibull exponential $\exp(-\hat{\beta}p^\eta)$ generates a sub- or super-exponential spectral cutoff for $0 < \eta < 1$ and $\eta > 1$, respectively [41–44]. In the following, we will consider the opposite nearly degenerate quantum limit [45,46] of density (2.5), $-\alpha \gg 1$, and rename the fugacity parameter as $\lambda = -\alpha$.

The electronic number count N, internal energy U and pressure P read

$$N = V\int_0^\infty \mathrm{d}\rho(p), \quad U = V\int_0^\infty E(p)\mathrm{d}\rho(p), \quad P = \frac{1}{3}\int_0^\infty E'(p)p\mathrm{d}\rho(p). \tag{2.6}$$

Since the partition function is related to the pressure by $\log Z = \beta PV$ in an equilibrium system, we use integration by parts to find

$$\log Z = \frac{4\pi sV}{(2\pi)^3}\int_0^\infty \log(1+\mathrm{e}^{-\alpha-\beta E(p)})p^2\mathrm{d}p, \tag{2.7}$$

which can also be obtained by a fermionic trace calculation [16,17]. The entropy is assembled as $S(\alpha,\beta,V) = \beta PV - \lambda N + \beta U$, with the integral representations (2.6) of particle count, energy and pressure substituted and $\lambda = -\alpha$, cf. after (2.5).

3. Quantifying dispersion in the nearly degenerate quantum regime: fugacity expansion

The integral representations (2.6) of the quantized thermodynamic variables are of type

$$I[g] = \int_0^\infty \frac{g(p)\mathrm{d}p}{\mathrm{e}^{F(p)-\lambda}+1}, \tag{3.1}$$

where $F(p) = \beta E(p) = \hat{\beta}p^\eta$, see after (2.5), and $g(p)$ is a power-law function. (We have put $\lambda = -\alpha$ and $\hat{\beta} = \beta/a^\eta$.) The integrals (4.1) can be evaluated by employing a Sommerfeld expansion valid for large $\lambda \gg 1$,

$$I[g] = \int_0^{F^{-1}(\lambda)} g(p)\mathrm{d}p + \frac{\pi^2}{6}g_F'(\lambda) + \frac{7\pi^4}{360}g_F^{(3)}(\lambda) + \mathrm{O}(g_F^{(5)}(\lambda)), \tag{3.2}$$

$$g_F(\lambda) = g(F^{-1}(\lambda))(F^{-1})'(\lambda), \tag{3.3}$$

where $F^{-1}(\lambda) = \lambda^{1/\eta}/\hat{\beta}^{1/\eta}$. The power-law functions $g(p)$ in (3.1) defining the particle count, internal energy and pressure can be read off from (2.6),

$$\frac{N}{V} = \frac{4\pi s}{(2\pi)^3}I[g_N(p) = p^2],$$

$$\frac{U}{V} = \frac{4\pi s}{(2\pi)^3}I[g_U(p) = p^{\eta+2}/a^\eta], \quad P = \frac{\eta}{3}\frac{U}{V}. \tag{3.4}$$

The entropy is thus $S(\alpha,\beta,V) = (1+\eta/3)\beta U - \lambda N$.

It suffices to calculate integral $I[g]$ in (3.1) with a power-law kernel $g(p) = p^\kappa$, $\kappa > 0$, so that $g_F(\lambda) = \lambda^{(1+\kappa-\eta)/\eta}/(\eta\hat{\beta}^{(1+\kappa)/\eta})$ as defined in (3.3). It will also be convenient to introduce the parameter $\xi = F^{-1}(\lambda)$ or inversely $\lambda = \hat{\beta}\xi^\eta$, and to express the derivatives $g_F'(\lambda)$ and $g_F^{(3)}(\lambda)$ arising in the Sommerfeld expansion (3.2) of $I[g(p) = p^\kappa]$ in terms of ξ:

$$g_F'(\lambda) = (1+\kappa-\eta)\frac{\xi^{1+\kappa}}{\eta^2(\hat{\beta}\xi^\eta)^2},$$

$$g_F^{(3)}(\lambda) = (1+\kappa-\eta)(1+\kappa-2\eta)(1+\kappa-3\eta)\frac{\xi^{1+\kappa}}{\eta^4(\hat{\beta}\xi^\eta)^4}. \tag{3.5}$$

The integral determining the leading order in expansion (3.2) gives $\xi^{\kappa+1}/(\kappa+1)$. The fugacity expansions of the number count and internal energy in (3.4) are found by specifying $\kappa = 2$ and $\kappa = 2+\eta$, respectively,

$$\frac{N}{V} = \frac{4\pi s}{(2\pi)^3}\frac{\xi^3}{3}\left[1 + \frac{\pi^2}{6}\frac{3(3-\eta)}{\eta^2(\hat{\beta}\xi^\eta)^2} + \frac{7\pi^4}{360}\frac{3(3-\eta)(3-2\eta)(3-3\eta)}{\eta^4(\hat{\beta}\xi^\eta)^4} + \cdots\right], \tag{3.6}$$

$$\frac{U}{V} = \frac{4\pi s}{(2\pi)^3}\frac{1}{a^\eta}\frac{\xi^{3+\eta}}{3+\eta}\left[1 + \frac{\pi^2}{6}\frac{(3+\eta)3}{\eta^2(\hat{\beta}\xi^\eta)^2} + \frac{7\pi^4}{360}\frac{(3+\eta)3(3-\eta)(3-2\eta)}{\eta^4(\hat{\beta}\xi^\eta)^4} + \cdots\right], \tag{3.7}$$

The expansions of the pressure and partition function are obtained via the internal energy expansion (3.7), since $P = \eta U/(3V)$ and $\log Z = \beta U\eta/3$, cf. (3.4). The entropy is assembled by substituting the series (3.6) and (3.7) into $S(\alpha,\beta,V) = (1+\eta/3)\hat{\beta}a^\eta U - \hat{\beta}\xi^\eta N$, see after (3.4). In this case, the leading orders in (3.6) and (3.7) cancel each other,

$$\frac{S}{V} = \frac{4\pi s}{(2\pi)^3}\frac{\xi^3}{3\eta}\frac{\pi^2}{\hat{\beta}\xi^\eta}\left[1 + \frac{7\pi^2}{30}\frac{(3-\eta)(3-2\eta)}{\eta^2(\hat{\beta}\xi^\eta)^2} + \cdots\right]. \tag{3.8}$$

The indicated second-order term is defined by the third-order terms in (3.6) and (3.7); apart from that, we will only need the second order in (3.6) and (3.7). We note that $\hat{\beta}\xi^{\eta} = \lambda = -\alpha$ is the fugacity parameter, cf. after (3.4), and $\eta > 0$ is the exponent of the electronic dispersion relation (2.4). $\hat{\beta} = \beta/a^{\eta}$ is the temperature parameter rescaled with the amplitude of the dispersion relation, see after (2.5). The expansions (3.6)–(3.8) give the thermodynamic variables in (α, β, V) representation; they are based on the ultra-relativistic dispersion relation $E(p) = p^{\eta}/a^{\eta}$ in (2.4) and apply in the low-temperature high-density regime, $\hat{\beta}\xi^{\eta} \gg 1$. In Section 4, we will identify ξ as density parameter, cf. (4.2).

4. Effect of electron dispersion on thermodynamic variables at low temperature and high density

4.1. Internal energy, entropy and isochoric heat capacity

We start by inverting the fugacity expansion (3.6) of the number density, solving for ξ. Using the shortcut

$$\hat{N} = \frac{N}{V}\frac{3(2\pi)^3}{4\pi s}, \tag{4.1}$$

we find

$$\xi(\beta, N, V) = \hat{N}^{1/3}\left(1 - \frac{\pi^2}{6}\frac{(3-\eta)}{\eta^2(\hat{\beta}\hat{N}^{\eta/3})^2} + \cdots\right). \tag{4.2}$$

The leading order thereof is the Fermi momentum, $p_F = \hat{N}^{1/3}$, and the ultra-relativistic Fermi energy/temperature is $E_F = k_B T_F = p_F^{\eta}/a^{\eta}$ with $a^{\eta} = m^{\eta-1}\varepsilon_0\mu_0$ according to the dispersion relation (2.4). Substituting $\xi(\beta, N, V)$ into the series expansions (3.7) and (3.8), we obtain the (T, N, V) parametrization of the internal energy, that is the caloric equation of state of the electron gas,

$$\frac{U}{V} = \frac{4\pi s}{(2\pi)^3 a^{\eta}}\frac{\hat{N}^{\eta/3+1}}{\eta+3}\left(1 + \frac{\pi^2}{6}\frac{\eta+3}{\eta(\hat{\beta}\hat{N}^{\eta/3})^2} + \cdots\right), \tag{4.3}$$

and the entropy $S(T, N, V)$,

$$\frac{S}{V} = \frac{4\pi s}{(2\pi)^3}\frac{1}{3\eta}\frac{\pi^2}{\hat{\beta}}\hat{N}^{(3-\eta)/3}\left(1 + \frac{\pi^2}{10}\frac{(2-3\eta)(3-\eta)}{\eta^2(\hat{\beta}\hat{N}^{\eta/3})^2} + \cdots\right). \tag{4.4}$$

The leading order thereof can be written as $S/N \sim \pi^2/(\eta\beta E_F)$, with Fermi energy $E_F = \hat{N}^{\eta/3}/a^{\eta}$. The expansion is in powers of $1/(\hat{\beta}\hat{N}^{\eta/3})^2$, where $\hat{\beta}\hat{N}^{\eta/3} = \beta E_F = T_F/T$. The leading order of the internal energy (4.3) is independent of temperature and defines the equipartition ratio $U/N \sim 3E_F/(3+\eta)$, cf. (4.1). The isochoric heat capacity $C_V = TS_{,T}$ reads like S in (4.4), with the second-order term multiplied by a factor of 3.

4.2. Mechanical and thermal stability: thermal equation of state, isobaric heat capacity, isobaric expansion coefficient, isothermal and adiabatic compressibility

The thermal equation of state $P(T, N, V)$ is obtained by substituting the Sommerfeld expansion (4.3) of the internal energy into the ultra-relativistic identity $P = \eta U/(3V)$, cf. (3.4),

$$\hat{P} = \hat{N}^{\eta/3+1}\left(1 + \frac{\pi^2}{6}\frac{\eta+3}{\eta(\hat{\beta}\hat{N}^{\eta/3})^2} + \cdots\right), \tag{4.5}$$

where we have introduced the rescaled pressure variable

$$\hat{P} = \frac{3(\eta+3)a^{\eta}}{\eta}\frac{(2\pi)^3}{4\pi s}P. \tag{4.6}$$

To obtain the volume in $V(T, P, N)$ parametrization, we solve (4.5) for V (which appears in the rescaled number density $\hat{N}$, cf. (4.1)),

$$V = \frac{3(2\pi)^3}{4\pi s}N\hat{P}^{-3/(\eta+3)}\left(1 + \frac{\pi^2}{2}\frac{1}{\eta(\hat{P}^{\eta/(\eta+3)}\hat{\beta})^2} + \cdots\right). \tag{4.7}$$

We substitute this into $S(T, V, N)$ in (4.4) to find the $S(T, P, N)$ representation of entropy,

$$S = \hat{P}^{-\eta/(\eta+3)}\frac{1}{\eta}\frac{\pi^2}{\hat{\beta}}N\left(1 + \frac{\pi^2}{30}\frac{(2\eta-3)(7\eta-6)}{\eta^2(\hat{\beta}\hat{P}^{\eta/(\eta+3)})^2} + \cdots\right). \tag{4.8}$$

The isobaric heat capacity $C_P(T, P, N) = TS_{,T}$ reads like S in (4.8), with the second-order term multiplied by a factor of 3. By substituting the thermal Eq. (4.5) into $C_P(T, P, N)$, we find the (T, V, N) parametrization of the isobaric heat capacity,

$$C_P = V\hat{N}^{1-\eta/3}\frac{4\pi s}{(2\pi)^3}\frac{1}{3\eta}\frac{\pi^2}{\hat{\beta}}\left(1 + \frac{\pi^2}{30}\frac{54 - 99\eta + 37\eta^2}{\eta^2(\hat{\beta}\hat{N}^{\eta/3})^2} + \cdots\right). \tag{4.9}$$

By comparing this with the isochoric heat capacity, see after (4.4), we find

$$C_P(T,V,N) \sim C_V + V\frac{\pi^4}{9}\frac{4\pi s}{(2\pi)^3}\frac{1}{\eta\hat{N}^{\eta-1}\hat{\beta}^3}. \tag{4.10}$$

The isothermal compressibility $\kappa_T = -V_{,P}/V$ and isobaric expansion coefficient $\alpha_{\rm exp} = V_{,T}/V$ are calculated from the series $V(T,P,N)$ in (4.7),

$$\kappa_T = \frac{3}{\eta+3}\frac{1}{P}\left(1+\frac{\pi^2}{3}\frac{1}{(\hat{P}^{\eta/(\eta+3)}\hat{\beta})^2}+\cdots\right), \quad \alpha_{\rm exp} \sim \frac{\pi^2 a^{2\eta}}{\eta\hat{P}^{2\eta/(\eta+3)}\beta}. \tag{4.11}$$

Using the leading order of the thermal equation in (4.5) and (4.6), we find

$$\frac{\alpha_{\rm exp}^2}{\beta\kappa_T} \sim \frac{\pi^4}{9}\frac{4\pi s}{(2\pi)^3}\frac{1}{\eta\hat{N}^{\eta-1}\hat{\beta}^3}, \tag{4.12}$$

so that the equilibrium relation $C_P = C_V + VT\alpha_{\rm exp}^2/\kappa_T$ is satisfied, cf. (4.10).

To obtain the adiabatic compressibility, $\kappa_S = -V_{,P}/V$, we need the $V(S,P,N)$ representation of the volume factor. To this end, we solve $S(T,P,N)$ in (4.8) for $\hat{\beta}$ in leading order, $\hat{\beta} \sim \pi^2 N/(\eta\hat{P}^{\eta/(3+\eta)}S)$, and substitute this into $V(T,P,N)$ in (4.7),

$$V(S,P,N) = \frac{3(2\pi)^3}{4\pi s}N\hat{P}^{-3/(\eta+3)}\left(1+\frac{\eta}{2\pi^2}\frac{1}{(N/S)^2}+\cdots\right). \tag{4.13}$$

The adiabatic compressibility is then found as $\kappa_S = 3/((\eta+3)P)$, and we note the ratios, cf. (4.10) and (4.11),

$$\frac{\kappa_T}{\kappa_S} = 1+\frac{\pi^2}{3}\frac{1}{(\hat{P}^{\eta/(\eta+3)}\hat{\beta})^2}+\cdots, \quad \frac{C_P}{C_V} = 1+\frac{\pi^2}{3}\frac{1}{(\hat{\beta}\hat{N}^{\eta/3})^2}+\cdots, \tag{4.14}$$

which are identical via the thermal Eq. (4.5), reflecting the general equilibrium relation $C_P/C_V = \kappa_T/\kappa_S$. The mechanical and thermal stability conditions $\kappa_T > \kappa_S > 0$ and $C_P > C_V > 0$ are thus satisfied for positive exponents η in the electronic dispersion relation (2.4).

4.3. Adiabatic bulk modulus, adiabatic incompressibility and speed of sound

We transform the caloric and thermal equations of state $U(T,N,V)$ and $P(T,N,V)$, cf. (4.3) and (4.5), into the adiabatic (N,V,S) representation by inverting the entropy variable in (4.4) in leading order,

$$\hat{\beta} = \frac{V}{\hat{S}}\hat{N}^{(3-\eta)/3}, \quad \hat{S} = \frac{3(2\pi)^3}{4\pi s}\frac{\eta}{\pi^2}S. \tag{4.15}$$

The pressure $P(N,V,S)$ then reads, cf. (4.5) and (4.6),

$$P = \frac{\eta}{\eta+3}\left(\frac{3(2\pi)^3}{4\pi s}\right)^{\eta/3}\frac{(\hbar c)^\eta}{(mc^2)^{\eta-1}\varepsilon_0\mu_0}\left(\frac{N}{V}\right)^{\eta/3+1}\left[1+\frac{1}{6\pi^2}\frac{\eta(\eta+3)}{(Nk_{\rm B}/S)^2}+\cdots\right], \tag{4.16}$$

where we used $a^\eta = m^{\eta-1}\varepsilon_0\mu_0$, cf. after (2.4), and $\hat{N}$ in (4.1), and we have restored the units. (The permeability amplitudes ε_0 and μ_0 are dimensionless.) The fugacity expansion of the internal energy is found as $U(N,V,S) = 3VP/\eta$ with expansion (4.16) substituted, cf. (3.4), so that $P = -U_{,V}(N,V,S)$.

The adiabatic bulk modulus $K_S = -VP_{,V}(N,V,S)$ (coinciding with the reciprocal compressibility $1/\kappa_S$, cf. after (4.12)) reads $K_S = (1+\eta/3)P$ with P in (4.16) substituted. Instead of volume, we may use the number density $n = N/V$ as parameter, so that $K_S = nP_{,n}(N,N/n,S)$ and

$$P = \frac{n^2}{N}U_{,n}(N,N/n,S), \quad P_{,n} = n\frac{\partial^2 nU(N,N/n,S)/N}{\partial n^2}, \tag{4.17}$$

where nU/N is the specific internal energy density and $P_{,n} = K_S/n = V^2U_{,V,V}(N,V,S)/N$ the adiabatic incompressibility (compression modulus).

As for the speed of sound, we consider the electron gas in an ionized medium of mass density $\rho = m_{\rm p}\mu_{\rm n}N/V$, where $m_{\rm p}$ is the proton mass and $\mu_{\rm n}$ the molecular weight per electron, see after (5.2). The squared sound velocity in the medium is then obtained as pressure derivative with respect to mass density, $\upsilon_{\rm s}^2 = P_{,\rho}(N, m_{\rm p}\mu_{\rm n}N/\rho, S) = K_S/\rho$. Thus, $\upsilon_{\rm s}^2 = (1+\eta/3)P/\rho$, where we can switch back to the (T,N,V) representation and substitute $P(T,N,V)$ in (4.5) and (4.6). Compression modulus and sound velocity are related by $P_{,n} = m_{\rm p}\mu_{\rm n}\upsilon_{\rm s}^2$.

Since $E_{\rm F} = k_{\rm B}T_{\rm F} = (p_{\rm F}/a)^\eta$ with $a^\eta = m^{\eta-1}\varepsilon_0\mu_0$ and $p_{\rm F} = \hat{N}^{1/3}$, $\hat{N} = \frac{3(2\pi)^3}{4\pi s}N/V$, see (4.1) and after (4.2), we can write Fermi energy and momentum as

$$k_{\rm B}T_{\rm F} = \frac{(\hbar c)^\eta}{(mc^2)^{\eta-1}\varepsilon_0\mu_0}\left(\frac{N}{V}\frac{3(2\pi)^3}{4\pi s}\right)^{\eta/3}, \quad p_{\rm F} = \hbar\left(\frac{N}{V}\frac{3(2\pi)^3}{4\pi s}\right)^{1/3}. \tag{4.18}$$

In the fugacity expansion of the rescaled pressure variable $\hat{P}$ in (4.5), we can express the electronic number density N/V (or $\hat{N}$ in (4.1)) by $T_{\rm F}$, using $\hat{\beta}\hat{N}^{\eta/3} = T_{\rm F}/T$ with $\hat{\beta} = \beta/a^{\eta}$. The fugacity expansion of the speed of sound $\upsilon_{\rm s}$ thus reads, cf. (4.1), (4.5) and (4.6),

$$\frac{\upsilon_{\rm s}^2}{c^2} = \frac{\eta+3}{3}\frac{P(T,N,V)}{m_{\rm p}c^2\mu_{\rm n}}\frac{V}{N} = \frac{\eta}{3}\frac{k_{\rm B}T_{\rm F}}{m_{\rm p}c^2\mu_{\rm n}}\left(1+\frac{\pi^2}{6}\frac{\eta+3}{\eta}\left(\frac{T}{T_{\rm F}}\right)^2+\cdots\right). \tag{4.19}$$

In Eq. (4.18), the number density N/V can be replaced by $\rho/(m_{\rm p}\mu_{\rm n})$, where ρ is the mass density of the ionized background medium, see after (4.17).

5. Effect of electron dispersion on the mass–radius relation of high-mass white dwarfs

The Newtonian hydrostatic equations of stellar structure read $P'(r) = -GM(r)\rho(r)/r^2$ and $M'(r) = 4\pi r^2\rho(r)$, where $\rho(r)$ is the mass density, $M(r)$ the mass in a sphere of radius r, $P(r)$ the pressure and G the gravitational constant. These equations can be combined to a second-order equation, $(r^2P'(r)/\rho(r))' = -4\pi Gr^2\rho(r)$. The boundary conditions are $\rho(0) = \rho_{\rm cent}$ (central density) and $\rho'(0) = 0$ (so that $\rho(r)$ does not have a cusp at the center). We assume a polytropic equation of state, that is a power-law relation between pressure and mass density, $P(r) = K\rho^{\gamma}(r)$ (which is the case for a totally degenerate ultra-relativistic electron gas in a dispersive medium, see Sections 3 and 4), to be substituted into the mentioned second-order equation. We write the exponent as $\gamma = 1+\eta/3$, to relate it to the thermal Eq. (4.16) of the electron gas in the zero-temperature limit; $\gamma = K_S/P$ is the isentropic expansion factor defined by bulk modulus and pressure, cf. after (4.16), and $P_{,\rho} = K(1+\eta/3)\rho^{\eta/3} = \upsilon_{\rm s}^2$ coincides with the leading order (zero-temperature limit) of the squared sound velocity in (4.19).

The above stated second-order equation can be transformed to Lane–Emden form, $(\hat{r}^2\theta'(\hat{r}))' + \hat{r}^2\theta^q(\hat{r}) = 0$, with polytropic index $q = 3/\eta$ and boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$, where $\rho(r) = \rho_{\rm cent}\theta^{3/\eta}(\hat{r})$ and $r = \sqrt{K(3+\eta)/(4\pi G\eta)}\rho_{\rm cent}^{(\eta-3)/6}\hat{r}$; $\rho_{\rm cent}$ is the central density $\rho(0)$. Solutions of the Lane–Emden equation are stable for $\eta > 1$ and unstable for $\eta < 1$. (The total gravitational potential energy is $-\eta U$, via the virial theorem with P related to the thermal energy U as in (3.4).) We denote the first zero of $\theta(\hat{r})$ by $\hat{r}_0$, which exists for stable solutions and defines the radius of the star according to the indicated variable transformations,

$$R = \sqrt{\frac{K(3+\eta)}{4\pi G\eta}}\rho_{\rm cent}^{(\eta-3)/6}\hat{r}_0. \tag{5.1}$$

The ratio K/G has the dimension $[{\rm g}^2{\rm cm}^{-4}({\rm g/cm}^3)^{-(3+\eta)/3}]$, and $\hat{r}_0$ and $\theta(\hat{r})$ are dimensionless. The total mass of the star is calculated as

$$M = 4\pi\int_0^R \rho(r)r^2{\rm d}r = 4\pi\left(\frac{K(3+\eta)}{4\pi G\eta}\right)^{3/2}\rho_{\rm cent}^{(\eta-1)/2}\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|, \tag{5.2}$$

where we used the Lane–Emden equation and integration by parts.

The mass density of a white dwarf is related to the electron density N/V in (3.6) by $\rho = m_{\rm p}\mu_{\rm n}N/V$, where $m_{\rm p}$ is the proton mass and $\mu_{\rm n}$ the nucleon–electron ratio (molecular weight per electron), usually $\mu_{\rm n} = A/Z \approx 2$ for white dwarfs. Using the zero-temperature (zero-entropy) limit of the thermal equation (4.16), we find $P = K\rho^{1+\eta/3}$ with amplitude K defined by

$$\frac{K(3+\eta)}{4\pi G\eta} = \frac{(\hbar c)^{\eta}}{(mc^2)^{\eta-1}}\frac{1}{4\pi G}\frac{1}{\varepsilon_0\mu_0}\left(\frac{3(2\pi)^3}{4\pi s}\right)^{\eta/3}\left(\frac{1}{m_{\rm p}\mu_{\rm n}}\right)^{(3+\eta)/3}, \tag{5.3}$$

where we have restored the units. No further units need to be restored in (5.1) and (5.2) other than substituting this ratio. m denotes the electron mass and $s = 2$ is the electronic spin degeneracy, cf. (2.5). The product $\varepsilon_0\mu_0$ of the permeability constants and the exponent η define the ultra-relativistic electronic dispersion relation (2.4).

Replacing the electronic number density by the stellar mass density $\rho = m_{\rm p}\mu_{\rm n}N/V$ in the Fermi momentum (4.18), we find

$$p_{\rm F} = \left(\frac{\rho}{m_{\rm p}\mu_{\rm n}}\right)^{1/3}\left(\frac{3(2\pi)^3}{4\pi s}\right)^{1/3}, \tag{5.4}$$

and inversely, $\rho = 4\pi sp_{\rm F}^3m_{\rm p}\mu_{\rm n}/(3(2\pi)^3)$. Since the ultra-relativistic limit (2.4) of the dispersion relation applies for $p_{\rm F}/m \gg 1$, it is convenient to define a critical mass density $\rho_{\rm crit}$ by putting $p_{\rm F} = m$ in (5.4), so that

$$\rho_{\rm crit} = \frac{4\pi s}{3(2\pi)^3}\frac{c^6}{(\hbar c)^3}m^3m_{\rm p}\mu_{\rm n}, \tag{5.5}$$

where we have restored the units. We note $\rho_{\rm crit} = 9.810\times10^5\mu_{\rm n}$ g/cm^3 and the ratio $\mu_{\rm n}\rho_{\odot}/\rho_{\rm crit} = 1.437\times10^{-6}$, with average solar density $\rho_{\odot} = 1.410$ g/cm^3. The nucleon–electron ratio is $\mu_{\rm n} \approx 2$, unless the mass density becomes high enough for neutronization to occur, which tends to reduce the electron number, see after (6.9). The ultra-relativistic regime is $\rho/\rho_{\rm crit} \gg 1$, since $p_{\rm F}/(mc) = (\rho/\rho_{\rm crit})^{1/3}$.

The stellar radius and mass in (5.1) and (5.2) can be rewritten as

$$\frac{R}{R_0} = \frac{\hat{r}_0}{\sqrt{\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0}}\left(\mu_{\rm n}\frac{\rho_{\rm cent}}{\rho_{\rm crit}}\right)^{(\eta-3)/6}, \tag{5.6}$$

$$\frac{M}{M_0} = \frac{\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|}{(\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0)^{3/2}}\left(\mu_{\rm n}\frac{\rho_{\rm cent}}{\rho_{\rm crit}}\right)^{(\eta-1)/2}, \tag{5.7}$$

where $\rho_{\rm cent}$ is the central mass density, and we introduced the shortcuts

$$R_0 = \left(\frac{3(2\pi)^3}{4\pi s}\right)^{1/2}\frac{(\hbar c)^{3/2}}{c^2}\frac{1}{\sqrt{4\pi G}}\frac{1}{mm_{\rm p}}, \tag{5.8}$$

$$M_0 = 4\pi\left(\frac{3(2\pi)^3}{4\pi s}\right)^{1/2}\frac{(\hbar c)^{3/2}}{(4\pi G)^{3/2}}\frac{1}{m_{\rm p}^2}, \tag{5.9}$$

defining a radius and mass scale, $R_0 = 7.713 \times 10^8$ cm and $M_0 = 5.657 \times 10^{33}$ g. For comparison, the solar radius and mass scales are $R_\odot = 6.957 \times 10^{10}$ cm and $M_\odot = 1.989 \times 10^{33}$ g, cf. Ref. [47]. We also note $GM_0/(R_0c^2) = m/m_{\rm p}$ and $4\pi\rho_{\rm crit} = \mu_{\rm n}M_0/R_0^3$, cf. (5.5).

Combining Eqs. (5.6) and (5.7) by eliminating $\rho_{\rm cent}/\rho_{\rm crit}$, we find the mass–radius relation

$$\frac{M}{M_0} = \frac{\hat{r}_0^{3(\eta-1)/(3-\eta)}\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|}{(\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0)^{3/(3-\eta)}}\left(\frac{R}{R_0}\right)^{3(1-\eta)/(3-\eta)}, \tag{5.10}$$

and inversely,

$$\frac{R}{R_0} = \frac{\hat{r}_0(\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|)^{(1-\eta/3)/(\eta-1)}}{(\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0)^{1/(\eta-1)}}\left(\frac{M}{M_0}\right)^{-(1-\eta/3)/(\eta-1)}. \tag{5.11}$$

We will consider exponents $1 < \eta < 3$ in the dispersion relation (2.4). In this interval, the mass increases and the radius decreases with increasing central density $\rho_{\rm cent}$, cf. (5.6) and (5.7). At the lower edge $\eta = 1$, the mass in (5.7) becomes independent of the central density, $M/M_0 = \hat{r}_0^2\left|\theta'(\hat{r}_0)\right|/(\mu_{\rm n}^{4/3}\varepsilon_0\mu_0)^{3/2}$, and the Chandrasekhar mass limit is recovered for vacuum permeabilities $\varepsilon_0 = \mu_0 = 1$. That is, the first zero $\hat{r}_0 = 6.8969$, $\hat{r}_0^2\theta'(\hat{r}_0) = -2.0182$ of the solution $\theta(\hat{r})$ of the Lane–Emden equation (with $\eta = 1$, $q = 3$) and the conversion $M_\odot/M_0 = 0.3515$ to solar units gives the mass limit $M/M_\odot = 5.74/\mu_{\rm n}^2$ or 1.435 for $\mu_{\rm n} = 2$, cf. e.g. Ref. [48]. As mentioned, $\eta = 1$ is the borderline between stable ($\eta > 1$) and unstable solutions of the Lane–Emden equation. At the upper edge $\eta = 3$, the radius (5.6) becomes independent of $\rho_{\rm cent}$, $R/R_0 = \hat{r}_0/\sqrt{\mu_{\rm n}^2\varepsilon_0\mu_0}$ with $\hat{r}_0 = \pi$. When discussing supernova progenitors in Section 6, we will use the polytropic index $q = 3/\eta = 2.4188$, that is $\eta = 1.2403$.

If mass and radius of the white dwarf are known as well as the exponent $1 < \eta < 3$ in dispersion relation (2.4), we can infer the product $\varepsilon_0\mu_0$ of the permeability constants from the mass–radius relation (5.10),

$$\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0 = \hat{r}_0^{\eta-1}(\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|)^{1-\eta/3}\left(\frac{R}{R_0}\right)^{1-\eta}\left(\frac{M}{M_0}\right)^{\eta/3-1}, \tag{5.12}$$

and the central mass density $\rho_{\rm cent}$ from (5.6) and (5.7),

$$\mu_{\rm n}\frac{\rho_{\rm cent}}{\rho_{\rm crit}} = \frac{\hat{r}_0^3}{\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|}\left(\frac{R}{R_0}\right)^{-3}\frac{M}{M_0}, \tag{5.13}$$

where the dimensionless constants $\hat{r}_0$ and $\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|$ only depend on the exponent η via the Lane–Emden equation. The average mass density $\rho_{\rm av} = 3M/(4\pi R^3)$ reads, in units of $\rho_{\rm crit}$ (see (5.5) and after (5.9)),

$$\mu_{\rm n}\frac{\rho_{\rm av}}{\rho_{\rm crit}} = 3\left(\frac{R}{R_0}\right)^{-3}\frac{M}{M_0}, \tag{5.14}$$

so that $\rho_{\rm cent}/\rho_{\rm av} = \hat{r}_0^3/(3\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|)$, cf. (5.13). Using the numerical values of $\hat{r}_0$ and $\hat{r}_0^2\theta'(\hat{r}_0)$ stated after (6.1), we find the ratio of central and average density of high-mass white dwarfs as $\rho_{\rm cent}/\rho_{\rm av} = 20.69$.

6. Super-Chandrasekhar mass white dwarfs

6.1. Mass, radii and central densities

We start with two high-mass white dwarfs, Sirius B and LHS 4033. Mass and radius estimates for Sirius B, obtained by combining parallax, surface temperature and gravitational redshift measurements [32] are $(0.94 \pm 0.05)M_\odot$ and $(8.4 \pm 2.5) \times 10^{-3}R_\odot$. A white dwarf even closer to the Chandrasekhar limit of 1.44 $M_\odot$ is LHS 4033, with $(1.33 \pm 0.02)M_\odot$ and $(3.6 \pm 0.2) \times 10^{-3}R_\odot$, cf. Ref. [33]. The conversion from solar units to M_0 and R_0 mass and radius scales, cf. (5.8) and (5.9), is effected by $M_\odot/M_0 = 0.3515$ and $R_\odot/R_0 = 90.20$.

We write the mass–radius relation (5.11) as power law $R/R_0 = A_0(M/M_0)^{-a}$, and determine the amplitude $A_0 = 5.074 \times 10^{-2}$ and exponent $a = 2.4414$ by substituting the quoted Sirius B and LHS 4033 values. By comparing with Eq. (5.11), we can infer the scaling exponent η of the dispersion relation (2.4) and the product $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0$, where ε_0 and μ_0 are permeability constants, cf. Section 2.1, and $\mu_{\rm n}$ is the nucleon–electron ratio,

$$\eta = \frac{a+1}{a+1/3}, \quad \mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0 = \frac{\hat{r}_0^{\eta-1}(\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|)^{1-\eta/3}}{A_0^{\eta-1}}. \tag{6.1}$$

We obtain $\eta = 1.2403$ and the polytropic index $q = 3/\eta = 2.4188$. The scaling exponent of the equation of state $P = K\rho^\gamma$ is $\gamma = 1 + 1/q = 1.4134$, cf. the beginning of Section 5. Specifying the polytropic index as $q = 2.4188$ in the Lane–Emden equation, we find the zero of the solution $\theta(\hat{r})$ located at $\hat{r}_0 = 5.1644$ and the derivative $\hat{r}_0^2\theta'(\hat{r}_0) = -2.2193$; the notation is defined at the beginning of Section 5. We then employ the second identity in (6.1) to calculate $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0 = 4.8467$. Using $\mu_{\rm n} = 2$, we can specify the permeability amplitudes and exponents defining the ultra-relativistic dispersion relation (2.4) as $\mu_0 = 1$, $\varphi = 0$ and $\chi = -0.2403$, $\varepsilon_0 = 1.8195$.

The central mass density is inferred from (5.7),

$$\mu_n\frac{\rho_{\rm cent}}{\rho_{\rm crit}} = \left(\frac{M}{M_0}\frac{(\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0)^{3/2}}{\hat{r}_0^2\left|\theta'(\hat{r}_0)\right|}\right)^{2/(\eta-1)}. \tag{6.2}$$

Switching to solar units and using the above numerical values for η, $\hat{r}_0$, $\hat{r}_0^2\theta'(\hat{r}_0)$ and $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0$, we find the mass–radius relation (5.11) as $R/R_\odot = 7.222 \times 10^{-3}(M/M_\odot)^{-2.4414}$ and the central density (6.2) as $\mu_{\rm n}\rho_{\rm cent}/\rho_{\rm crit} = 78.92(M/M_\odot)^{8.3242}$.

Remark. We have obtained the empirical mass–radius relation $R/R_0 = A_0(M/M_0)^{-a}$ using Sirius B and LHS 4033 data points. If more mass–radius data points above 1 $M_\odot$ become available, one can determine the amplitude A_0 and the exponent a by a least-squares fit. The scaling exponent η of the electronic dispersion relation (2.4) is then obtained as stated in (6.1). For the individual mass–radius data points used in the fit, one can calculate the product $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0$ individually via (5.12) and the central densities via (5.13). If only a mass estimate of the star is available, as it happens for white dwarf progenitors of supernovae, one uses the amplitude A_0 obtained from the least-squares fit to calculate $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0$ via (6.1) (as done here) and the central density via (6.2). The radii of the progenitors are estimated by means of the mass–radius relation obtained from the least-squares fit. Since both Sirius B and LHS 4033 are located on the $R/R_0 = A_0(M/M_0)^{-a}$ curve (determined in this way), the estimate of $\mu_{\rm n}^{1+\eta/3}\varepsilon_0\mu_0$ coincides for both stars and can be calculated via (5.12) or (6.1) and their central densities via (5.13) or (6.2).

The possibility of white dwarfs exceeding the Chandrasekhar mass limit is suggested by several Type Ia supernovae (SNe Ia) with mass ejecta substantially above the mass limit of 1.44 $M_\odot$, cf. after (5.11). An ejecta mass of 1.6 $M_\odot$ was estimated for SN 2013cv in Ref. [1] and of 2.1 $M_\odot$ for SN 2003fg in Ref. [2] and of 2.4 $M_\odot$ for SN 2007if in Ref. [3]. The mass ejecta of SN 2009dc were estimated to be 2.8 $M_\odot$, cf. Ref. [4], the highest mass estimate of a super-Chandrasekhar mass SN Ia obtained so far.

To obtain estimates of the radii of the white dwarf progenitors of SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc, we use the mass–radius relation derived from Sirius B and LHS 4033, see after (6.2), and the quoted mass estimates, see Table 1. Also listed in Table 1 are the average mass density $\rho_{\rm av} = 3M/(4\pi R^3)$ in solar units as well as normalized with the critical density $\rho_{\rm crit}$, cf. (5.14), and the central density $\rho_{\rm cent}$, also in solar units and normalized with $\rho_{\rm crit}$, calculated as stated after (6.2).

6.2. Electronic number density, Fermi momentum and temperature

The Fermi momentum is related to the mass density by $p_{\rm F} = mc\mu_n^{-1/3}(\mu_n\rho/\rho_{\rm crit})^{1/3}$, cf. after (5.5). By making use of $p_{\rm F}(N/V)$ and $E_{\rm F}(N/V)$ in (4.18), we find the $\rho/\rho_{\rm crit}$ parametrization of the electronic number density and Fermi energy,

$$n = \frac{N}{V} = \frac{4\pi s}{3(2\pi)^3}\frac{(mc^2)^3}{(\hbar c)^3\mu_n}\frac{\mu_n\rho}{\rho_{\rm crit}}, \tag{6.3}$$

Table 1
Mass, radius and density parameters of Sirius B and LHS 4033 and of the white dwarf progenitors of the super-Chandrasekhar mass supernovae SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc. The mass and radius estimates of Sirius B and LHS 4033 are taken from Refs. [32,33] and the progenitor masses from Refs. [1–4]. The progenitor radii are obtained by applying the dispersive mass–radius relation (5.11) with permeability constants inferred from the high-mass white dwarfs Sirius B and LHS 4033, cf. Section 6.1. Also recorded are the average and central mass densities ρ_{av} and ρ_{cent} in solar units ($\rho_\odot = 1.410$ g/cm^3) and in units of the critical density $\rho_{crit} = 9.810 \times 10^5 \mu_n$ g/cm^3 (see (5.5)) which defines the ultra-relativistic regime $\rho/\rho_{crit} \gg 1$. μ_n denotes the molecular weight per electron (nucleon–electron ratio).

	$M/M_\odot$	$R/R_\odot$	$\rho_{av}/\rho_\odot$	$\rho_{cent}/\rho_\odot$	$\mu_n\rho_{av}/\rho_{crit}$	$\mu_n\rho_{cent}/\rho_{crit}$
Sirius B	0.94 ± 0.05	$(8.4 \pm 2.5) \times 10^{-3}$	1.59×10^6	3.28×10^7	2.28	47.2
LHS 4033	1.33 ± 0.02	$(3.6 \pm 0.2) \times 10^{-3}$	2.85×10^7	5.90×10^8	41.0	848
progen. SN 2013cv	1.6	2.29×10^{-3}	1.33×10^8	2.75×10^9	191	3.95×10^3
progen. SN 2003fg	2.1	1.18×10^{-3}	1.28×10^9	2.64×10^{10}	1.84×10^3	3.80×10^4
progen. SN 2007if	2.4	8.52×10^{-4}	3.88×10^9	8.03×10^{10}	5.58×10^3	1.15×10^5
progen. SN 2009dc	2.8	5.85×10^{-4}	1.40×10^{10}	2.90×10^{11}	2.01×10^4	4.16×10^5

Table 2
Electronic number density, Fermi momentum/energy/temperature, gravitational surface potential and surface gravity of Sirius B, LHS 4033 and the super-Chandrasekhar mass progenitors of SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc. The electron density $n(\rho) = N/V$ and Fermi momentum, energy and temperature $p_F(\rho)$, $E_F(\rho)$, $T_F(\rho)$ (see (6.3) and (6.4)) are calculated at the average and central mass densities ρ_{av} and ρ_{cent} listed in Table 1. These quantities scale with the nucleon–electron ratio, $n \propto 1/\mu_n$, $p_F \propto \mu_n^{-1/3}$, $E_F \propto T_F \propto \mu_n$, since the fit parameter $\mu_n^{1+\eta/3}\varepsilon_0\mu_0 = 4.85$ is kept fixed, see Section 6.1, and they are listed here for $\mu_n = 2$. (The permeability amplitudes ε_0 and μ_0 and the scaling exponent η define the electronic dispersion relation (2.4).) The surface potential $GM/(c^2R)$ and surface gravity GM/R^2, cf. after (6.9), are based on the mass and radius estimates in Table 1.

	$n(\rho_{av})$ [cm^{-3}]	$n(\rho_{cent})$ [cm^{-3}]	$p_F(\rho_{av})$ [MeV/c]	$p_F(\rho_{cent})$ [MeV/c]	$E_F(\rho_{av})$ [MeV]	$E_F(\rho_{cent})$ [MeV]	$T_F(\rho_{av})$ [10^{10} K]	$T_F(\rho_{cent})$ [10^{10} K]	$GM/(c^2R)$	GM/R^2 [cm/s^2]
Sirius B	6.68×10^{29}	1.38×10^{31}	0.534	1.47	0.297	1.03	0.344	1.20	2.38×10^{-4}	3.65×10^8
LHS 4033	1.20×10^{31}	2.49×10^{32}	1.40	3.84	0.983	3.42	1.14	3.97	7.84×10^{-4}	2.81×10^9
progen. SN 2013cv	5.60×10^{31}	1.16×10^{33}	2.34	6.41	1.85	6.47	2.15	7.51	1.48×10^{-3}	8.37×10^9
progen. SN 2003fg	5.38×10^{32}	1.11×10^{34}	4.97	13.6	4.72	16.5	5.47	19.1	3.78×10^{-3}	4.14×10^{10}
progen. SN 2007if	1.64×10^{33}	3.39×10^{34}	7.19	19.7	7.46	26.1	8.66	30.3	5.98×10^{-3}	9.07×10^{10}
progen. SN 2009dc	5.90×10^{33}	1.22×10^{35}	11.0	30.3	12.7	44.4	14.7	51.5	1.02×10^{-2}	2.24×10^{11}

Table 3
Electron degeneracy pressure P, bulk modulus K_S, compression modulus (volume incompressibility) $P_{,n}$ and speed of sound υ_s. These quantities depend on the mass density ρ, cf. (6.5), (6.7) and (6.8), and are listed for the average and central densities ρ_{av} and ρ_{cent} (see Table 1). The compression modulus linearly scales with the molecular weight per electron, $P_{,n} \propto \mu_n$, and is recorded here for $\mu_n = 2$, see the remarks after (6.8) and the caption to Table 2. The pressure is proportional to the bulk modulus, $P = 0.7075\,K_S$, cf. (6.7), due to the polytropic equation of state; the conversion to cgs pressure units is 1 MeV/cm^3 = 1.602×10^{-6} dyn/cm^2.

	$P(\rho_{av})$ [dyn/cm^2]	$P(\rho_{cent})$ [dyn/cm^2]	$K_S(\rho_{av})$ [MeV/cm^3]	$K_S(\rho_{cent})$ [MeV/cm^3]	$P_{,n}(\rho_{av})$ [MeV]	$P_{,n}(\rho_{cent})$ [MeV]	$\upsilon_s(\rho_{av})/c$	$\upsilon_s(\rho_{cent})/c$
Sirius B	9.28×10^{22}	6.72×10^{24}	8.19×10^{28}	5.93×10^{30}	0.123	0.429	8.08×10^{-3}	1.51×10^{-2}
LHS 4033	5.51×10^{24}	3.99×10^{26}	4.86×10^{30}	3.52×10^{32}	0.405	1.42	1.47×10^{-2}	2.75×10^{-2}
progen. SN 2013cv	4.85×10^{25}	3.50×10^{27}	4.28×10^{31}	3.09×10^{33}	0.764	2.67	2.02×10^{-2}	3.77×10^{-2}
progen. SN 2003fg	1.19×10^{27}	8.60×10^{28}	1.05×10^{33}	7.59×10^{34}	1.95	6.82	3.22×10^{-2}	6.03×10^{-2}
progen. SN 2007if	5.72×10^{27}	4.14×10^{29}	5.05×10^{33}	3.65×10^{35}	3.09	10.8	4.05×10^{-2}	7.58×10^{-2}
progen. SN 2009dc	3.50×10^{28}	2.54×10^{30}	3.09×10^{34}	2.24×10^{36}	5.24	18.3	5.29×10^{-2}	9.89×10^{-2}

$$E_F = k_B T_F = \frac{\mu_n m c^2}{\mu_n^{1+\eta/3}\varepsilon_0\mu_0}\left(\frac{\mu_n\rho}{\rho_{crit}}\right)^{\eta/3}, \tag{6.4}$$

where the nucleon–electron ratio μ_n has been scaled into the equations. ($\mu_n\rho/\rho_{crit}$ is independent of μ_n since $\rho_{crit} \propto \mu_n$, cf. (5.5).) Inserting the numerical values for η and $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$ as stated after (6.1), we obtain the number density $n = 5.865 \times 10^{29}\mu_n^{-1}(\mu_n\rho/\rho_{crit})$ cm^{-3}.

Converting to solar units using $\mu_n\rho_\odot/\rho_{crit} = 1.437 \times 10^{-6}$, cf. after (5.5), we can write $p_F = 5.767 \times 10^{-3}\mu_n^{-1/3}(\rho/\rho_\odot)^{1/3}$ MeV/c and $n = 8.430 \times 10^{23}\mu_n^{-1}(\rho/\rho_\odot)$ cm^{-3}. We also note the ratios $p_F(\rho_{cent})/p_F(\rho_{av}) = (\rho_{cent}/\rho_{av})^{1/3} = 2.745$ and $n(\rho_{cent})/n(\rho_{av}) = \rho_{cent}/\rho_{av} = 20.69$, where ρ_{cent} and ρ_{av} are the central and average mass densities, see after (5.14). Analogously, the Fermi temperature reads $T_F = 1.223 \times 10^9\mu_n(\mu_n\rho/\rho_{crit})^{0.4134}$ K or $T_F = 4.701 \times 10^6\mu_n(\rho/\rho_\odot)^{0.4134}$ K in solar units, and $T_F(\rho_{cent})/T_F(\rho_{av}) = (\rho_{cent}/\rho_{av})^{\eta/3} = 3.50$.

The Fermi momentum scales with the nucleon–electron ratio as $p_F \propto \mu_n^{-1/3}$. The number density and Fermi energy $E_F(\rho)$[MeV] $= 0.8617\,T_F(\rho)[10^{10}$ K] scale as $n \propto \mu_n^{-1}$ and $E_F \propto \mu_n$, since the fitting parameter $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$ is kept fixed. In Table 2, we list $n(\rho_{av,cent})$, $p_F(\rho_{av,cent})$, $E_F(\rho_{av,cent})$ and $T_F(\rho_{av,cent})$ for $\mu_n = 2$.

Remark. Quantum gravity effects are believed to become relevant at the Planck energy scale $E_P = c^2\sqrt{\hbar c/G} = 1.2 \times 10^{19}$ GeV. Since the central Fermi energy $E_F(\rho_{cent})$ of high-mass white dwarfs is in the MeV range, see Table 2, one expects such effects to be negligible in white dwarfs. It has recently been suggested that quantum gravity could be manifested at a lower energy scale E_P/α_P, where α_P is a large dimensionless constant yet to be determined [49,50]. For instance, the central Fermi energy of the SN 2009dc progenitor is 44 MeV, which thus requires $\alpha_P \sim 3\times 10^{20}$ for quantum gravity effects to emerge. A similar scale factor $\alpha_P \sim 10^{21}$ has been used in Ref. [51] to obtain a noticeable decrease of the Chandrasekhar mass as a quantum gravity effect. That is, a rescaling of the Planck scale by a factor of this order of magnitude is needed for quantum gravity effects to become observable in white dwarfs. Quantum gravity corrections to Lamb shifts and to the muon anomalous magnetic moment have been calculated in Refs. [49,50], where an upper bound $\alpha_P < 10^{10}$ was obtained by comparison with high-precision measurements of the ground-state Lamb shift in hydrogen, and an even tighter bound $\alpha_P < 10^8$ was inferred from muon $g-2$ experiments.

6.3. Speed of sound in high-mass white dwarfs, their compression modulus and gravitational surface potential

The $\rho/\rho_{\rm crit}$ parametrization of the sound velocity υ_s in (4.19) (zero-temperature limit thereof) is obtained by substituting k_BT_F in (6.4) into (4.19),

$$\frac{\upsilon_s^2}{c^2} = \frac{\eta}{3}\frac{m}{m_p}\frac{1}{\mu_n^{1+\eta/3}\varepsilon_0\mu_0}\left(\frac{\mu_n\rho}{\rho_{\rm crit}}\right)^{\eta/3}. \tag{6.5}$$

Using the numerical values for η and $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$, cf. after (6.1), and the electron–proton mass ratio, we find $\upsilon_s/c = 6.816\times 10^{-3}(\mu_n\rho/\rho_{\rm crit})^{0.2067}$ or $\upsilon_s/c = 4.225\times 10^{-4}(\rho/\rho_\odot)^{0.2067}$ in solar units, and the ratio $\upsilon_s(\rho_{\rm cent})/\upsilon_s(\rho_{\rm av}) = (\rho_{\rm cent}/\rho_{\rm av})^{\eta/6} = 1.87$. We also note the mass and radius scaling of the squared speed of sound at the center,

$$\frac{\upsilon_s^2(\rho_{\rm cent})}{c^2} = \frac{\eta}{3}\frac{m}{m_p}\frac{\hat r_0}{\hat r_0^2\left|\theta'(\hat r_0)\right|}\left(\frac{R}{R_0}\right)^{-1}\frac{M}{M_0}, \tag{6.6}$$

where we made use of $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$ in (5.12) and $\mu_n\rho_{\rm cent}/\rho_{\rm crit}$ in (5.13).

The $\rho/\rho_{\rm crit}$ parametrization of the bulk modulus K_S and compression modulus $P_{,n}$ (see Section 4.3) is found by substituting the number density $n = N/V$ in (6.3) into the pressure variable P in (4.16) (with the entropy dependent correction term dropped at zero temperature/entropy) and using the critical density $\rho_{\rm crit}$ in (5.5),

$$K_S = \left(1+\frac{\eta}{3}\right)P = \frac{\eta}{3}\frac{4\pi s}{3(2\pi)^3}\frac{(mc^2)^4}{(\hbar c)^3}\frac{1}{\mu_n^{1+\eta/3}\varepsilon_0\mu_0}\left(\frac{\mu_n\rho}{\rho_{\rm crit}}\right)^{1+\eta/3}, \tag{6.7}$$

$$P_{,n} = \frac{K_S}{n} = \frac{\eta}{3}\frac{\mu_n mc^2}{\mu_n^{1+\eta/3}\varepsilon_0\mu_0}\left(\frac{\mu_n\rho}{\rho_{\rm crit}}\right)^{\eta/3}. \tag{6.8}$$

Inserting the numerical values as above (see after (6.1)), we find the bulk modulus $K_S = 2.557\times 10^{28}(\mu_n\rho/\rho_{\rm crit})^{1.4134}$ MeV/cm^3 and the compression modulus (incompressibility) $P_{,n} = 4.359\times 10^{-2}\mu_n(\mu_n\rho/\rho_{\rm crit})^{0.4134}$ MeV. In solar units, $K_S = 1.412\times 10^{20}(\rho/\rho_\odot)^{1.4134}$ MeV/cm^3 and $P_{,n} = 1.675\times 10^{-4}\mu_n(\rho/\rho_\odot)^{0.4134}$ MeV. The ratios of these variables taken at the central and average mass densities read $K_S(\rho_{\rm cent})/K_S(\rho_{\rm av}) = (\rho_{\rm cent}/\rho_{\rm av})^{1+\eta/3} = 72.39$ and $P_{,n}(\rho_{\rm cent})/P_{,n}(\rho_{\rm av}) = (\rho_{\rm cent}/\rho_{\rm av})^{\eta/3} = 3.50$, cf. after (5.14).

The compression modulus is related to the speed of sound and the Fermi energy by $P_{,n} = m_p\mu_n\upsilon_s^2 = (\eta/3)E_F$, see after (4.17), (6.4) and (6.5). $P_{,n}$ scales linearly with the molecular weight per electron, $P_{,n} \propto \mu_n$, in contrast to the sound velocity and the bulk modulus which do not scale with μ_n since the parameter $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$ is kept fixed, cf. after (6.1). The pressure, the bulk and compression moduli and the speed of sound are listed in Table 3, evaluated at the central and average densities of Sirius B, LHS 4033 and the progenitor white dwarfs of the supernovae SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc.

The gravitational surface potential is related to the speed of sound at the center by, cf. after (5.9) and (6.6),

$$\frac{GM}{c^2R} = \frac{m}{m_p}\frac{M}{M_0}\left(\frac{R}{R_0}\right)^{-1} = \frac{3}{\eta}\frac{\hat r_0^2\left|\theta'(\hat r_0)\right|}{\hat r_0}\frac{\upsilon_s^2(\rho_{\rm cent})}{c^2}, \tag{6.9}$$

and can be calculated with the mass and radius estimates in Table 1, just by rescaling the solar potential, $GM/(c^2R) = 2.123\times 10^{-6}(R/R_\odot)^{-1}M/M_\odot$. The surface potential of the super-Chandrasekhar mass white dwarfs recorded in Table 2 is still small enough for the Newtonian limit of the gravitational redshift to apply, $z \sim GM/(c^2R)$. Using (6.9) and the numerical values for η, $\hat r_0$ and $\hat r_0^2\theta'(\hat r_0)$ stated after (6.1), we find $\upsilon_s^2(\rho_{\rm cent})/c^2 = 0.9620\,GM/(Rc^2)$. The surface gravity is $GM/R^2[{\rm cm/s^2}] = 2.742\times 10^4(R/R_\odot)^{-2}M/M_\odot$, see Table 2.

If the central density $\rho_{\rm cent}$ approaches the neutron drip density $\rho_{\rm drip} = 4.3\times 10^{11}{\rm g/cm^3}$, cf. Ref. [52], as is the case for the progenitors of SN 2007if and SN 2009dc, see Table 1, neutronization due to electron capture (via inverse beta decay, $e^- + p \to n + \nu_e$) can increase the nucleon–electron ratio from $\mu_n \approx 2$ up to $\mu_n \approx 3.3$, depending on the chemical composition of the white dwarf. The number density, Fermi momentum and Fermi energy of the progenitors of SN 2007if and SN 2009dc in Table 2 are therefore upper/lower bounds (calculated at $\mu_n = 2$), to be rescaled by a factor of $(2/\mu_n)^{1/3}$,

$2/\mu_n$ and $\mu_n/2$, respectively, see after (6.4). The radius and density estimates in Table 1 are unaffected by an increase of μ_n, since the product $\mu_n^{1+\eta/3}\varepsilon_0\mu_0$ of the nucleon–electron ratio and the permeability amplitudes is kept fixed as a fitting parameter in the mass–radius relation, so that a larger μ_n is compensated by smaller permeability amplitudes $\varepsilon_0\mu_0$ in the dispersion relation (2.4). The speed of sound and the bulk modulus in Table 3 are also unaffected by an increasing μ_n for the same reason, whereas the compression modulus in Table 3 increases by a factor of $\mu_n/2$.

7. Conclusion

The recently observed super-Chandrasekhar mass thermonuclear supernovae [1–4] suggest the existence of white dwarf progenitor stars with masses above the Chandrasekhar limit of 1.44 $M_\odot$. We have found a thermal equation of state for the ultra-relativistic electron gas in high-mass white dwarfs which takes account of the permeability of the ionized stellar matter constituting the neutralizing background of the electron plasma. The ionized medium is manifested by a permeability tensor in the electronic Dirac equation leading to a nonlinear dispersion relation $E \sim m(p/m)^\eta/(\varepsilon_0\mu_0)$ for ultra-relativistic electrons, in contrast to the linear vacuum relation $E \sim p$, see Section 2.1. The power-law index η and the product $\varepsilon_0\mu_0$ of the permeability amplitudes are determined empirically.

In the zero-temperature limit, the equation of state of the ultra-relativistic degenerate plasma pervading the dispersive medium admits a polytropic form, $P = K\rho^{1+\eta/3}$, where ρ is the stellar mass density proportional to the electron density N/V, η is the power-law exponent of the electronic dispersion relation, and the proportionality constant is inversely proportional to the product of the permeability amplitudes, $K \propto 1/(\varepsilon_0\mu_0)$, cf. (5.3). As the equation of state is polytropic, the stellar structure equations can be reduced to a Lane–Emden equation, which admits stable solutions leading to a genuine mass–radius relation for dispersion indices in the interval $1 < \eta < 3$, see (5.10) and (5.11). In contrast, constant (vacuum) permeabilities result in a linear ultra-relativistic dispersion relation (with $\eta = 1$), and the mass–radius relation degenerates into a limit mass.

In Section 6, we employed the dispersive mass–radius relation (5.11) together with mass and radius estimates of two high-mass white dwarfs, Sirius B and LHS 4033, cf. Table 1, to infer the scaling exponent $\eta = 1.240$ and the product $\varepsilon_0\mu_0 = 4.85/\mu_n^{1+\eta/3}$ of the permeability amplitudes defining the ultra-relativistic electronic dispersion relation (2.4); μ_n is the nucleon–electron ratio. By specifying these constants in the mass–radius relation (5.11) and using estimates of the mass ejecta of the super-Chandrasekhar mass Type Ia supernovae SN 2013cv, SN 2003fg, SN 2007if and SN 2009dc, we obtained estimates of the radii of their white dwarf progenitors, cf. Table 1. We also found estimates of the sound velocity in the progenitor stars, as well as of their central mass density, Fermi temperature and bulk and compression moduli, cf. Tables 1–3. In the case of supernova SN 2009dc, the mass of the progenitor star is about 2.8 $M_\odot$, almost twice the Chandrasekhar limit mass, and the central density is reaching the neutron drip density $\rho_{drip}/\rho_\odot = 3.05 \times 10^{11}$, cf. Table 1.

In Sections 5 and 6, we considered a totally degenerate dispersive electron gas at zero temperature, which suffices to derive the mass–radius relation of high-mass white dwarfs above the Chandrasekhar mass limit. In Sections 3 and 4, we discussed the effect of nonlinear electron dispersion in the nearly degenerate ultra-relativistic regime. We derived the low-temperature high-density fugacity expansions of the thermodynamic variables, in particular their dependence on the scaling exponent η of the electronic dispersion relation, and demonstrated the mechanical and thermal stability, $\kappa_T > \kappa_S > 0$ and $C_P > C_V > 0$, of an ultra-relativistic low-temperature plasma in a dispersive medium.

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